Master Thesis[*]

RDF, Risk Dynamics into the future[†]

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Contents

Abstract

In this paper, I give an introduction to the *RDF(Risk Dynamics into the Future)* developed by AIS Group. The *RDF* method is a tool to help managers to determine the risk under certain economic scenarios. And my work focused on improving the numerical method to calculate the loss distribution. I presented three methods: Monte Carlo simulations, Nascent Delta method and a direct Saddlepoint Approximation method. Comparison of the numerical results are also included for some simple examples.

1 Introduction

The financial crisis since 2007 has shown the importance of risk management in today's financial world. The crisis which was triggered by a liquidity shortfall in the United State Banking System, has resulted in the collapse of large financial institutions, the "bail out" of banks by national governments and downturns in stock markets around the world. It is considered to be the worst financial crisis since the Great Depression in 1930s[1].

The collapse of a global housing bubble also caused the value of securities tied to the real estate to fall steeply. The Investor confidence was damaged, causing large losses in the stock markets. It is commonly understood that the credit rating agencies failed to price the risk with the mortgage-related financial product and the government did not adjust their regulatory practices to address 21st century financial markets[2].

These all shows the importance of risk management. A good understanding of potential risk is vital to many companies. So we seek ways to quantify risks. Credit risk management is risk assessment that comes in an investment. Risk often comes in investing and in the allocation of capital. The risks must be assessed so as to derive a sound investment decision. And decisions should be made by balancing the risks and returns. The risk of losses that result in the default of payment of the debtors is a kind of risk that must be expected. A bank to keep substantial amount of capital to protect its solvency and to maintain its economic stability. The greater the bank is exposed to risks, the greater the amount of capital must be when it comes to its reserves, so as to maintain its solvency and stability.

[1] Three top economists agree 2009 worst financial crisis since great depression; risks increase if right steps are not taken. REUTERS
http://www.reuters.com/article/pressRelease/idUS193520+27-Feb-2009+BW20090227

[2] Declaration of G20, Whitehouse
http://georgewbush-whitehouse.archives.gov/news/releases/2008/11/20081115-1.html

2 Background

Here, I will briefly introduce the terminologies and the AIS Group where I conducted this work.

2.1 Definitions and Theoretical Background

2.1.1 Expected Loss

In probability theory, the attribute *expected* always refers to an *expectation* or *mean value*, and this is also the case in risk management. The basic ideas is: The bank assigns to each obligator a *default probability* (DP), a loss fraction called the *loss given default* (LGD), describing the fraction of the loan's exposure expected to be lost in case of default, and the *exposure at default* (EAD) subject to be lost in the considered time period. The loss of any obligor is then defined by a *loss variable*

$$\tilde{L} = EAD \times LGD \times L \qquad with \qquad L = 1_D, \quad \mathbb{P}(D) = DP \tag{1}$$

where D denotes the *event* that the obligor defaults in a certain period of time (most often one year), and $\mathbb{P}(D)$ denotes the probability of D.The model is in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, consisting of a *sample space* Ω, a σ-Algebra $\mathcal{F}$, and a probability measure $\mathbb{P}$.

In this setting it very natural to define the *expected loss*(EL) of any customer as the expectation of its corresponding loss variable $\tilde{L}$, namely

$$EL = \mathbb{E}[\tilde{L}] = EAD \times LGD \times \mathbb{P}(D) = EAD \times LGD \times DP \tag{2}$$

To obtain representation (2) of the EL, we need some additional assumption on the constituents of Formula (1), for example, the assumption that EAD and LGD are constant values. This is not necessarily the case under all circumstances. There are various random variable due to uncertainties in amortization, usage, and other drivers of EAD up to the chosen planning horizon. In such cases the EL is still given by Equation (2) if one can assume that the exposure, the loss given default,and the default event D are independent and EAD & LGD are the expectations of some underlying random variables. But even the independence assumption is questionable and in general very much simplifying.

2.1.2 Ratings

Basically ratings describe the *creditworthiness* of customers. Hereby quantitative as well as qualitative information is used to evaluate a client. In practice, the rating procedure is often more based on the judgement and experience of the rating analyst than on pure mathematical procedures with strictly defined outcomes. It turns out that in the US and Canada, most issuers of public debt are rated at least by two of the three main rating agencies Moody's, S&P, and Fitch. Their reports on *corporate bond defaults* are publicly available, either at local offices or for web access.

2.1.3 Unexpected Loss

Holding capital as a cushion against *expected loss* is not enough. In fact, the bank should also save money for covering *unexpected losses* exceeding the average experienced losses from past history. As a measure of the magnitude of the deviation of losses from the EL, the standard deviation of the loss variable $\tilde{L}$ as defined in (1) is a natural choice. This quantity is called the *Unexpected Loss*(UL), defined by

$$UL = \sqrt{\mathbb{V}[\tilde{L}]} = \sqrt{\mathbb{V}[EAD \times \mathrm{SEV} \times L]} \tag{3}$$

2.1.4 Value at Risk

We seek others ways to quantify risk capital, hereby taking a *target level of statistical confidence* into account. If c is the selected confidence level, VaR corresponds to the 1-c lower-tail level. For example, for a 95 percent confidence level, the probability of loss less or equal to VaR is 95%

2.2 RDF(*Risk Dynamics into the Future*)

2.2.1 About AIS Group

Headquartered in Barcelona, Spain, AIS, Aplicaciones de Inteligencia Artificial, S.A. is a multinational with over 20 years experience, specializing in the development of automatic systems to help companies in decision-making processes.

The activities of AIS are threefold: consultancy, software design and creation of statistical and optimization models. Its solutions include: quantitative credit risk assessment, application of quantitative techniques in marketing, risk control in the insurance sector, optimisation of paper sizes, monitoring of production, consumption forecasts and optimum distribution of energy, cash and daily publications. It also offers expert consultancy in the different divisions. The activities of AIS in credit risk analysis in the financial sector have always been its core business.

2.2.2 Introduction to the RDF method

The RDF(Risk Dynamics into the future) method, developed by AIS, is an open project, allowing, by way of sophisticated econometric models, the simulation of unfavourable economic scenarios, the calculation of the distribution of losses in these scenarios and support for strategic planning and business development. It provides a new method for the calculation of stress testing on the economic capital of credit risk.

3 Main Problem

One criteria that the RDF method use is that the sources of variability which produce the risk result essentially from the macroeconomic situation.

3.1 Macroeconomics Model

The logit of the PD is a linear combination (a linear regression) of macroeconomic variables. The notation used is below. In the logit model, the probability of default of an obligor in sector i is related to the sector creditworthiness index,

$$P_i(X_t) = \frac{1}{1 + \exp(-(\vec{\beta}_i^T(B) \cdot X_t + \gamma_{i,t}))} \tag{4}$$

$$\begin{aligned} X_t &= \text{vector of macroeconomic variables} \\ \gamma_{i,t} &= \text{Error of the linear regression model of the portfolio } i = 1, ..m \text{ and time } t. \\ \vec{\beta}_i(B) &= \text{vector of polynomials of lag operator} \end{aligned}$$

3.2 Loss function

An important fact used for computation is that, conditional on a scenario, obligor defaults are independent. In most cases, a Monte Carlo simulation can be applied to determine portfolio conditional losses. But we want more effective computational tools. If a portfolio contains a very large number of obligors, each with a small marginal contribution, then the Law of Large Numbers can be applied to estimate the conditional portfolio losses. As the number of obligors approaches infinity, the conditional loss distribution converges to the mean losses over the scenario. Hence the total loss function are given by the sum of the expected losses of each portfolio.

$$\begin{aligned} L(X_t) &= \sum_{i=1,m} P_i(X_t)K_i = \sum_{i=1,m} \frac{K_i}{1+\exp(-(\vec{\beta}_i^T(B)\cdot X_t+\gamma_{i,t}))} \\ &= \sum_{i=1,m} \frac{K_i}{1+\exp(-(\bar{S}_{i,t}))} \end{aligned}$$

$$\begin{aligned} X_t &= \text{vector of macroeconomic variables} \\ K_i &= EAD_i \times LGD_i \text{ for each portfolio } i=1,..m \\ \gamma_{i,t} &= \text{Error of the linear regression model of the portfolio } i=1,..m \text{ and time } t. \\ \bar{S}_{i,t} &= \vec{\beta}_i^T(B)\cdot X_t+\gamma_{i,t} \\ \vec{\beta}_i(B) &= \text{vector of polynomial of lag operator} \end{aligned}$$

3.3 Calculating the loss distribution

In order to calculate the distribution of loss, we use the dirac delta function to capture the variable values corresponding to a certain level of loss as follows:

$$P\left(Y=y\right)=\iiint\limits_{X:\,X\in\Re^n}\Omega\left(X\right)\delta\left(y-L\left(X\right)\right)d^nX \tag{5}$$

$$\begin{aligned} &P(\,) && \textit{Loss probability density} \\ &Y && \textit{Loss} \\ &L(X) && \textit{Loss function with respect to the risk indicators} \\ &\Omega(X) && \textit{Joint probability distribution of risk indicators} \\ &\delta(s) && \textit{Delta de Dirac: if } \delta(x)=+\infty \textit{ at } 0, \textit{and is } 0 \textit{ elsewhere. With the property} \\ & && \int_{-\infty}^{+\infty}\delta(x)=1 \end{aligned}$$

3.4 Methods

3.4.1 Monte Carlo Simulations

Monte Carlo Simulation is very easy but sometime it is very time-consuming for a high level of accuracy. The method is just to generate the random variables according to their distributions, usually Multivariate normal distribution, like:

$$X \sim N(M\ \Sigma) \tag{6}$$

where M is the vector of mean values and Σ is the variance-covariance matrix. And for each realization of X, compute the loss

$$Y = L(X) \tag{7}$$

Then we analysis the density of Y and calculate the relevant quantities like VaR and Expected Loss. The number of simulations we used in the tests are 10,000,000

3.4.2 Delta Nascent Method

One main difficulty in the integral is the delta function. So we try to use different ways to calculate or remove the delta function. Here, the first method is to approximate it. The delta function can be viewed as the limit of a sequence of functions

$$\delta(x) = \lim_{a \to 0+} \delta_a(x) \tag{8}$$

where $\delta_a(x)$ is sometimes called a *nascent delta* function. This limit is meant in a weak sense:

$$\lim_{a \to 0+} \int_{-\infty}^{+\infty} \delta_a(x) f(x) dx = f(0) \tag{9}$$

for all smooth functions with compact support.
Here we will use the Gaussian kernel as the *nascent delta*.

$$\delta_a\left(x\right) = \frac{1}{\sqrt{2\pi}\,a} e^{-\frac{x^2}{2\,a^2}} \tag{10}$$

Then to approximate the probability density function $P(\cdot)$ of the loss Y, we use

$$P_a\left(Y = y\right) = \iiint\limits_{X:\, X \in \Re^n} \Omega\left(X\right) \delta_a\left(y - L\left(X\right)\right) d^n X \tag{11}$$

as

$$\underset{a \to 0}{Lim}\, P_a\left(Y = y\right) = P\left(Y = y\right) \tag{12}$$

Now, we have change the problem into choosing the parameter a for nascent delta and then calculate the approximated $P_a(\cdot)$.

First, we assume that we have found a good parameter a, then I will show how to calculate the $P_a(\cdot)$.

$$
\begin{aligned}
P_a(y) &= \iiint_X \frac{e^{-\frac{(y-L(X))^2}{2a^2}}}{\sqrt{2\pi}a} \Omega(X)\, dX \\
&= \iiint_X \frac{e^{-\frac{(y-L(X))^2}{2a^2}}}{\sqrt{2\pi}a} \frac{e^{-\frac{1}{2}(X-M)^{Tr}.\Sigma^{-1}.(X-M)}}{(2\pi)^{\frac{n}{2}}|\Sigma|^{\frac{1}{2}}} dX \\
&= \frac{1}{a(2\pi)^{\frac{n+1}{2}}|\Sigma|^{\frac{1}{2}}} \iiint_X e^{-\frac{(y-L(X))^2}{2a^2} - \frac{1}{2}(X-M)^{Tr}\Sigma^{-1}(X-M)} dX
\end{aligned}
$$

Now we consider the exponent part

$$
Elip(X,y) = -\frac{(y-L(X))^2}{2a^2} - \frac{1}{2}(X-M)^{Tr}\Sigma^{-1}(X-M) \tag{13}
$$

We use the second order Taylor expansion around a fixed point X_0, then

$$
\begin{aligned}
Elip(X,y) &\approx Elip(X_0,y) + Elip'(X_0,y)(X-X_0) \\
&\quad + \frac{1}{2}(X-X_0)^{Tr} Elip''(X_0,y)(X-X_0)
\end{aligned} \tag{14}
$$

And we want to eliminate the first order term. Notice that the *Elip* function is differentiable, so for fixed y, the local maxima has the property that makes the first order derivative zero. i.e.

$$
\frac{\partial Elip(X_{0,y},y)}{\partial X} = \frac{y-L(X_{0,y})}{a^2} L'(X_{0,y}) - (X_{0,y}-\mu)^{Tr}\Sigma^{-1} = 0 \tag{15}
$$

The calculation of $X_{0,y}$ has to be done on each value of y(amout of losses), we could use Newton's Method or other Quasi-Newton Methods. In the implementation, we used BFGS method, one of Quasi-Newton Methods.

Then, equation (14) becomes

$$
\begin{aligned}
Elip(X,y) &\approx Elip(X_{0,y},y) + \frac{1}{2}(X-X_{0,y})^{Tr}.Elip''(X_{0,y},y).(X-X_{0,y}) \\
&= -\frac{1}{2}\frac{(y-L(X_{0,y}))^2}{a^2} - \frac{1}{2}(X_{0,y}-M)^{Tr}\Sigma^{-1}(X_{0,y}-M) + \frac{1}{2}(X-X_{0,y})^{Tr}. \\
&\quad \left[\frac{y-L(X_{0,y})}{a^2}L''(X_{0,y}) - \frac{1}{a^2}L'(X_{0,y}).L'(X_{0,y})^{Tr} - \Sigma^{-1}\right].(X-X_{0,y})
\end{aligned}
$$

And

$$P_a(y) = \frac{1}{(2\pi)^{\frac{1}{2}} a} \frac{1}{(2\pi)^{\frac{n}{2}} |\Sigma|^{\frac{1}{2}}} e^{-\frac{\left(y-L\left(X_{0,y}\right)\right)^2}{2a^2} - \frac{1}{2}(X_{0,y}-M)^{Tr}\Sigma^{-1}(X_{0,y}-M)}$$

$$\iiint_X e^{-\frac{1}{2}(X-X_{0,y})^{Tr}\left[\Sigma^{-1} - \frac{y-L\left(X_{0,y}\right)}{a^2} L''(X_{0,y}) + \frac{1}{a^2} L'(X_{0,y}).L'(X_{0,y})^{Tr}\right](X-X_{0,y})} dX$$

Now we can calculate the integral analytically using the quadratic formula, like

$$\iiint_X \exp\left(-\frac{1}{2}(x-M)^{Tr}\Lambda(x-M)\right) dX = \frac{(2\pi)^{n/2}}{|\Lambda|^{1/2}} \tag{16}$$

And in our case

$$\Lambda = \Sigma^{-1} - \frac{y - L(X_{0,t})}{a^2} L''(X_{0,y}) + \frac{1}{a^2} L'(X_{0,y}).L'(X_{0,y})^{Tr} \tag{17}$$

Then, we have

$$P_a(y) = \frac{e^{-\frac{\left(y-L\left(X_{0,y}\right)\right)^2}{2a^2} - \frac{1}{2}(X_{0,y}-M)^{Tr}\Sigma^{-1}(X_{0,y}-M)}}{(2\pi)^{\frac{1}{2}} a \left| I - \frac{y-L(X_{0,y})}{a^2}\Sigma.L''(X_{0,y}) + \frac{1}{a^2}\Sigma.L'(X_{0,y}).L'(X_{0,y})^{Tr}\right|^{\frac{1}{2}}} \tag{18}$$

3.4.3 Saddlepoint Approximation

I was mainly focusing on the following method, saddlepoint approximation with lagrangian multiplier

$$\int_{X:\Re^n} \delta(y - f[X]) \frac{e^{-\frac{1}{2}(X-M)^{Tr}\Sigma^{-1}(X-M)}}{|\Sigma|^{\frac{1}{2}}(2\pi)^{\frac{n}{2}}} dX^n \tag{19}$$

First, we introduce X_y is such a point that, $y = f[X_y]$ being of and having the maximum value of probability density , a Lagrangian will be formed to calculate it.

$$\underset{\substack{X_y:Max\,Elip[X] \\ s.t.:y=f[X_y]}}{LagElip[X,\lambda]} = -\tfrac{1}{2}(X-M)^{Tr} \cdot \Sigma^{-1} \cdot (X-M) - \lambda\,(y - f[X])$$

With first derivatives, with respect to X and λ

$$\frac{\partial LagElip}{\partial X} = -(X-M)^{Tr} \cdot \Sigma^{-1} + \lambda f_X{}'^{Tr}[X]$$
$$\frac{\partial LagElip}{\partial \lambda} = y - f[X]$$

We are looking for the critical points such that make both equation equal zero, i.e.

$$\begin{array}{l} -(X_y-M)^{Tr} \cdot \Sigma^{-1} + \lambda\, f_X{}'^{Tr}[X_y] \;=0 \\ y - f[X_y] = 0 \end{array} \tag{20}$$

We introduced an efficient numerical method to compute the saddlepoint in *Appendix A*, which is an iterative method.

Now multiplying the first equation in (20) by $(X - X_y)$, we have

$$\lambda\, f_X{}'^{Tr}[X_y] \cdot (X-X_y) = (X_y-M)^{Tr} \cdot \Sigma^{-1} \cdot (X-X_y) \tag{21}$$

Then if we do

$$\begin{array}{rcl} elip[X] & = & -\frac{1}{2}(X-M)^{Tr} \cdot \Sigma^{-1} \cdot (X-M) \\ & = & -\frac{1}{2}(X_y-M)^{Tr} \cdot \Sigma^{-1} \cdot (X_y-M) - (X_y-M)^{Tr} \cdot \Sigma^{-1} \cdot (X-X_y) \\ & & -\frac{1}{2}(X-X_y)^{Tr} \cdot \Sigma^{-1} \cdot (X-X_y) \end{array}$$

Substitute using (21)

$$\begin{array}{rcl} elip[X] & = & -\frac{1}{2}(X-M)^{Tr} \cdot \Sigma^{-1} \cdot (X-M) \\ & = & -\frac{1}{2}(X_y-M)^{Tr} \cdot \Sigma^{-1} \cdot (X_y-M) \\ & & -\lambda\, f_X{}'^{Tr}[X] \cdot (X-X_y) - \frac{1}{2}(X-X_y)^{Tr} \cdot \Sigma^{-1} \cdot (X-X_y) \end{array}$$

Also, we develop Taylor expansion of the loss function at X_y

$$y - f[X] \cong y - f[X_y] - f'_X[X_y].(X-X_y)$$

Now come back to the original integral (19), we have

$$P^*(y) = \int\limits_{X:\Re^n} \delta\left(-f'_X{}^{Tr}[X_y](X-X_y)\right) \frac{e^{-\frac{1}{2}(X_y-M)^{Tr}\cdot\Sigma^{-1}\cdot(X_y-M)-\lambda\, f'_X{}^{Tr}[X_y]\cdot(X-X_y)-\frac{1}{2}(X-X_y)^{Tr}\cdot\Sigma^{-1}\cdot(X-X_y)}}{|\Sigma|^{\frac{1}{2}}(2\pi)^{\frac{n}{2}}} dX^n \tag{22}$$

Now, we will perform a change of variable in order to be able to integrate the dirac delta function. First, we write

$$X = \begin{pmatrix} v \\ V \end{pmatrix} \quad X_y = \begin{pmatrix} v_y \\ V_y \end{pmatrix}$$

$$f'_X(X_y) = \begin{pmatrix} f'_v \\ f'_V \end{pmatrix} \quad \Sigma = \begin{pmatrix} s_{vv} & s_{vV} \\ s_{Vv} & s_{VV} \end{pmatrix} \quad \Sigma^{-1} = \begin{pmatrix} r_{vv} & r_{vV} \\ r_{Vv} & r_{VV} \end{pmatrix}$$

And make the change of variable

$$\left(f'_v, {f'_V}^{Tr}\right) \cdot \begin{pmatrix} v - v_y \\ V - V_y \end{pmatrix} \to u$$

and keep the rest (n-1) variables the same. Then

$$v - v_y = \frac{u}{f'_v} - \frac{1}{f'_v} f'^{Tr}_V \cdot (V - V_y)$$

And the Jacobian Matrix is:

$$J = \nabla \begin{pmatrix} u \\ V \end{pmatrix} = \begin{pmatrix} f'_v & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

If we write $C = X_y - M$, the integral (22) becomes

$$\int_V \int_u \delta(u) \frac{e^{-\frac{1}{2}\mathrm{C}^{\mathrm{Tr}} \cdot \Sigma^{-1} \cdot \mathrm{C} - \lambda \left(f'_v, {f'_V}^{Tr}\right) \cdot \begin{pmatrix} \frac{u}{f'_v} - \frac{1}{f'_v} {f'_V}^{Tr} (V - V_y) \\ V - V_y \end{pmatrix}}}{|\Sigma|^{\frac{1}{2}} (2\pi)^{\frac{n}{2}} |J|}$$

$$\frac{e^{-\frac{1}{2}\left(\frac{u}{f'_v} - \frac{1}{f'_v}(V - V_y) f'_V, (V - V_y)^{Tr}\right) \Sigma^{-1} \cdot \begin{pmatrix} \frac{u}{f'_v} - \frac{1}{f'_v} {f'_V}^{Tr} (V - V_y) \\ V - V_y \end{pmatrix}}}{} du\, dV^{n-1}$$

$$\int_V \int_u \delta(u) \frac{e^{-\frac{1}{2} C^{Tr} \Sigma^{-1} C - \lambda\, u - \frac{1}{2}\left(\frac{u}{f_v{}'} - \frac{1}{f_v{}'}(V - V_y) f'_V, (V - V_y)^{Tr}\right) \Sigma^{-1} \cdot \begin{pmatrix} \frac{u}{f_v{}'} - \frac{1}{f_v{}'} {f'_V}^{Tr} \cdot (V - V_y) \\ V - V_y \end{pmatrix}}}{|\Sigma|^{\frac{1}{2}} (2\pi)^{\frac{n}{2}} |J|} du\, dV^{n-1} \tag{23}$$

Now we can integrate the dirac delta function inside the integral first.

$$\int_{V \in \Re^{n-1}} \delta(u) \frac{e^{-\frac{1}{2} C^{Tr} \Sigma^{-1} C - \frac{1}{2}\left(-\frac{1}{f'_v}(V - V_y) f'_V, (V - V_y)^{Tr}\right) \Sigma^{-1} \begin{pmatrix} -\frac{1}{f'_v} {f'_V}^{Tr} (V - V_y) \\ V - V_y \end{pmatrix}}}{|\Sigma|^{\frac{1}{2}} (2\pi)^{\frac{n}{2}} |J|} dV^{n-1}$$

$$(24)$$

Rearrange, we have

$$\int\limits_{V:\Re^{n-1}} \frac{e^{-\frac{1}{2}C^{Tr}\Sigma^{-1}C-\frac{1}{2}(V-V_y)^{Tr}\cdot\left(\begin{pmatrix} -\frac{1}{f'_v}{f'}_V & I \end{pmatrix}\cdot\begin{pmatrix} r_{vv} & r_{vV} \\ r_{Vv} & r_{VV} \end{pmatrix}\cdot\begin{pmatrix} -\frac{1}{f'_v}{{f'}_V}^{Tr} \\ I \end{pmatrix}\right)\cdot(V-V_y)}}{|\Sigma|^{\frac{1}{2}}(2\pi)^{\frac{n}{2}}|J|} dV^{n-1} \tag{25}$$

Then

$$\int\limits_{V:\Re^{n-1}} \frac{e^{-\frac{1}{2}C^{Tr}\Sigma^{-1}C-\frac{1}{2}(V-V_y)^{Tr}\cdot\left(\frac{1}{{f'_v}^2}r_{vv}{f'}_V\cdot{{f'}_V}^{Tr}-\frac{1}{f'_v}\left(r_{Vv}\cdot{{f'}_V}^{Tr}+{f'}_V\cdot r_{vV}\right)+r_{VV}\right)\cdot(V-V_y)}}{|\Sigma|^{\frac{1}{2}}(2\pi)^{\frac{n}{2}}|J|} dV^{n-1} \tag{26}$$

Now the integral takes a quadratic form, we can calculate it and obtain the following result.

$$\frac{e^{-\frac{1}{2}C^{Tr}\cdot\Sigma^{-1}\cdot C}}{(2\pi)^{\frac{1}{2}}|\Sigma|^{\frac{1}{2}}\left|\frac{1}{{f'_v}^2}r_{vv}{f'}_V\cdot{{f'}_V}^{Tr}-\frac{1}{f'_v}\left(r_{Vv}\cdot{{f'}_V}^{Tr}+{f'}_V\cdot r_{vV}\right)+r_{VV}\right|^{\frac{1}{2}}|J|} \tag{27}$$

$$\begin{aligned}
|J| &= |f'_v| \text{ the absolute value of the determinant of the Jacobian Matrix} \\
C &= X_y - M \\
X_y &= \text{Vector on } \Re^{\text{n}} \text{ defines the Saddlepoint.} \\
\sigma &= \text{Covariance Matrix} \\
|Z| &= \text{Determinant of Z if Z is a square matrix} \\
&\quad\ \text{Absolute Value of r if r is a scalar} \\
(v, V) &= \text{X, parts of X} \\
\begin{pmatrix} r_{vv} & r_{vV} \\ r_{Vv} & r_{VV} \end{pmatrix} &= \Sigma^{-1}\text{Blocks of inverse of covariance matrix}
\end{aligned}$$

Remark We need to point out that the matrix

$$\frac{1}{{f'_v}^2}r_{vv}{f'}_V\cdot{{f'}_V}^{Tr}-\frac{1}{{f_v}'}\left(r_{Vv}\cdot{{f'}_V}^{Tr}+{f'}_V\cdot r_{vV}\right)+r_{VV} \tag{28}$$

need to be positive definite in order to have rational square root. But it is not trivial to see this in such a complex form.

So this innovate us to find a method with more compact form. Before the change of variables, first, we perform a transform Z = U(X - $\mathrm{X_y}$) i.e. $\mathrm{U^{Tr}Z}$ = (X - $\mathrm{X_y}$) such that U is a unitary matrix and $\Sigma = U^{Tr}DU$ where D is a diagonal matrix of the eigenvalues of Matrix , this could be done by Singular Value Decomposition or Eigendecomposition.

Notice that $f'_X[X_y]$ and $X_y - \mathrm{M}$ are just constants and we denote $C = \mathrm{X_y} - M$. So from equation (22), we have

$$\int\limits_{Z:\Re^n} \delta\left(-{f'_X}^{Tr}[X_y].\mathrm{U^{Tr}Z}\right) \frac{e^{-\frac{1}{2}\mathrm{C}^{Tr}\Sigma^{-1}\mathrm{C}-\lambda\, {f'_X}^{Tr}[X_y]\,\mathrm{U^{Tr}Z}-\frac{1}{2}\mathrm{Z}^{Tr}D^{-1}\mathrm{Z}}}{|\Sigma|^{\frac{1}{2}}(2\pi)^{\frac{n}{2}}} \mathrm{abs}(\left|U^{Tr}\right|)dZ \tag{29}$$

Because U is an unitary matrix , then $\left|U^{Tr}\right| = \pm 1$, so abs($\left|U^{Tr}\right|$) is 1.

Now, we rewrite the variables

$$Z \to \begin{pmatrix} \mathrm{z} \\ Z_Z \end{pmatrix} \qquad U \cdot f'_X[X_y] \to \begin{pmatrix} t \\ T \end{pmatrix}$$

$$D \to \begin{pmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{pmatrix} \qquad D^{-1} \to \begin{pmatrix} d_1^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n^{-1} \end{pmatrix}$$

Then the equation (29) becomes

$$\iint\limits_{\substack{Z_Z:\Re^{n-1} \\ z:\Re^1}} \delta\left(-\left(t, T^{Tr}\right)\begin{pmatrix} \mathrm{z} \\ Z_Z \end{pmatrix}\right) \frac{e^{-\frac{1}{2}\mathrm{C}^{Tr}\Sigma^{-1}\mathrm{C}-\lambda\left(t,T^{Tr}\right)\begin{pmatrix} \mathrm{z} \\ Z_Z \end{pmatrix}-\frac{1}{2}\begin{pmatrix} \mathrm{z} \\ Z_Z \end{pmatrix}^{Tr}D^{-1}\cdot\begin{pmatrix} \mathrm{z} \\ Z_Z \end{pmatrix}}}{|\Sigma|^{\frac{1}{2}}(2\pi)^{\frac{n}{2}}} dz dZ_Z$$

Change the expression inside the dirac,

$$\left(t, T^{Tr}\right)\cdot\begin{pmatrix} \mathrm{z} \\ Z_Z \end{pmatrix} \to u \quad \mathrm{z} = \frac{u}{\mathrm{t}} - \frac{1}{\mathrm{t}}T^{Tr}\, Z_Z \quad Jacobian = \begin{pmatrix} t & & T^{Tr} & \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$|Jacobian| = \mathrm{t}$$

Then we have

$$\int\limits_{Z_Z\in\Re^{n-1}} \int\limits_{\mathrm{u}\in\Re^1} \delta(u) \frac{e^{-\frac{1}{2}C^{Tr}\Sigma^{-1}C-\lambda\,\mathrm{u}-\frac{1}{2}\left(\left({Z_Z}^{Tr},1\right)\begin{pmatrix} -\frac{T}{t} & I \\ \frac{u}{t} & 0 \end{pmatrix}\right)D^{-1}\left(\begin{pmatrix} -\frac{T^{Tr}}{t} & \frac{u}{t} \\ I & 0 \end{pmatrix}\begin{pmatrix} Z_Z \\ 1 \end{pmatrix}\right)}}{|\Sigma|^{\frac{1}{2}}(2\pi)^{\frac{n}{2}}\,|\mathrm{t}|} du\, dZ_Z$$

Now, we can integrate out the dirac delta,

$$\int_{Z_Z \in \Re^{n-1}} \frac{e^{-\frac{1}{2}C^{Tr}\cdot\Sigma^{-1}\cdot C - \frac{1}{2}\left(Z_Z{}^{Tr}\cdot\begin{pmatrix} -\frac{T}{t} & I \end{pmatrix}\right)\cdot\begin{pmatrix} d_1^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n^{-1} \end{pmatrix}\cdot\left(\begin{pmatrix} -\frac{T^{Tr}}{t} \\ I \end{pmatrix}\cdot Z_Z\right)}}{|\Sigma|^{\frac{1}{2}}(2\pi)^{\frac{n}{2}}\,|\mathrm{t}|} dZ_Z$$

$$= \int_{Z_Z \in \Re^{n-1}} \frac{e^{-\frac{1}{2}C^{Tr}\cdot\Sigma^{-1}\cdot C - \frac{1}{2}Z_Z{}^{Tr}\left(\frac{d_1^{-1}}{t^2}\cdot T\cdot T^{Tr} + \begin{pmatrix} d_2^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n^{-1} \end{pmatrix}\right)Z_Z}}{|\Sigma|^{\frac{1}{2}}(2\pi)^{\frac{n}{2}}\,|\mathrm{t}|} dZ_Z$$

Integrating as a quadratic form and simplifying

$$\frac{e^{-\frac{1}{2}C^{Tr}\cdot\Sigma^{-1}\cdot C}}{(2\pi)^{\frac{1}{2}}|\Sigma|^{\frac{1}{2}}\,|t|\left|\frac{d_1^{-1}}{t^2}\cdot T\cdot T^{Tr} + \begin{pmatrix} d_2^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n^{-1} \end{pmatrix}\right|^{\frac{1}{2}}}$$

$$\begin{aligned}
X_y &= \text{Vector on } \Re^{\mathrm{n}} \text{ defines the Saddlepoint.} \\
& \text{Lagrangean in this point is zero} \\
C &= X_y - M \\
\Sigma &= \text{Covariances Matrix} \\
&= U^{Tr}DU, where \text{ D is diagonal matrix and U is unitary matrix} \\
& \text{ i.e. Singular Value Decomposition of } \Sigma. \\
|Z| &= \text{Determinant of Z if Z is a square matrix} \\
& \text{Absolute Value of r if r is a scalar} \\
\begin{pmatrix} t \\ T \end{pmatrix} &= U{f'_X}^{Tr}[X_y] \\
t &= U(1,\cdot)\cdot{f'_X}^{Tr}[X_y] \\
T &= U(2:n,\cdot)\cdot{f'_X}^{Tr}[X_y] \\
C &= X_{\mathrm{y}} - M
\end{aligned}$$

Remark: $d_1^{-1}TT^{Tr} + t^2\begin{pmatrix} d_2^{-1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n^{-1} \end{pmatrix}$ is positive definite.

4 Numerical tests and conclusion

4.1 Examples

Here we will use two simple examples to illustrate the advantages and disadvantages of different methods.

4.1.1 An example of two segments with two variables

We consider an example of two segments with two variables. In this case, we are not able to obtain the loss distribution analytically, so we will compare the results from Monte Carlo Simulations, Nascent Delta Method and Direct Method

$$\mathrm{y} = \mathrm{g}\left(\mathrm{x}\right) = \frac{10000000}{1+e^{-\mathrm{x}_1}} + \frac{8000000}{1+e^{-\mathrm{x}_2}}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \sim N\left(\begin{pmatrix} -0.1 \\ -0.2 \end{pmatrix}, \begin{pmatrix} 0.2 & 0.1 \\ 0.1 & 1.3 \end{pmatrix}\right)$$

4.1.2 Another example of one segment with two variables

Here we consider an example of one segment with two variables.

$$\mathrm{y} = \mathrm{g}\left(\mathrm{x}\right) = \frac{10000000}{1+e^{-\mathrm{x}_1-\mathrm{x}_2}}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \sim N\left(\begin{pmatrix} -0.1 \\ -0.2 \end{pmatrix}, \begin{pmatrix} 0.2 & 0.1 \\ 0.1 & 0.3 \end{pmatrix}\right)$$

In this case, the good news is we could obtain the analytical solution as:

- First, let $z = x_1 + x_2$, find the distribution of z.

$$\begin{aligned} mean(z) &= mean(x_1) + mean(x_2) = 0.5 + 0.6 = 1.1 \\ Var(z) &= Var(x_1) + Var(x_2) + 2Cov(x_1, x_2) = 0.9 + 1.2 + 2{*}0.8 = 3.7 \end{aligned}$$

- Then the loss function becomes $\mathrm{y} = \mathrm{g}\left(\mathrm{z}\right) = \frac{10000000}{1+e^{-z}}$, we could compute the distribution of y analytically.

$$\begin{aligned} pdf_Y(y) &= \frac{pdf_X(x)}{|g_x(x)|} = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} / (10^6 e^{-x} / (1+e^{-x})^2) \\ &= \frac{10^{-6}}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} e^x (1+e^{-x})^2 \end{aligned}$$

  where $y = g(x)$ i.e. $x = \log(\frac{y}{10^6 - y})$

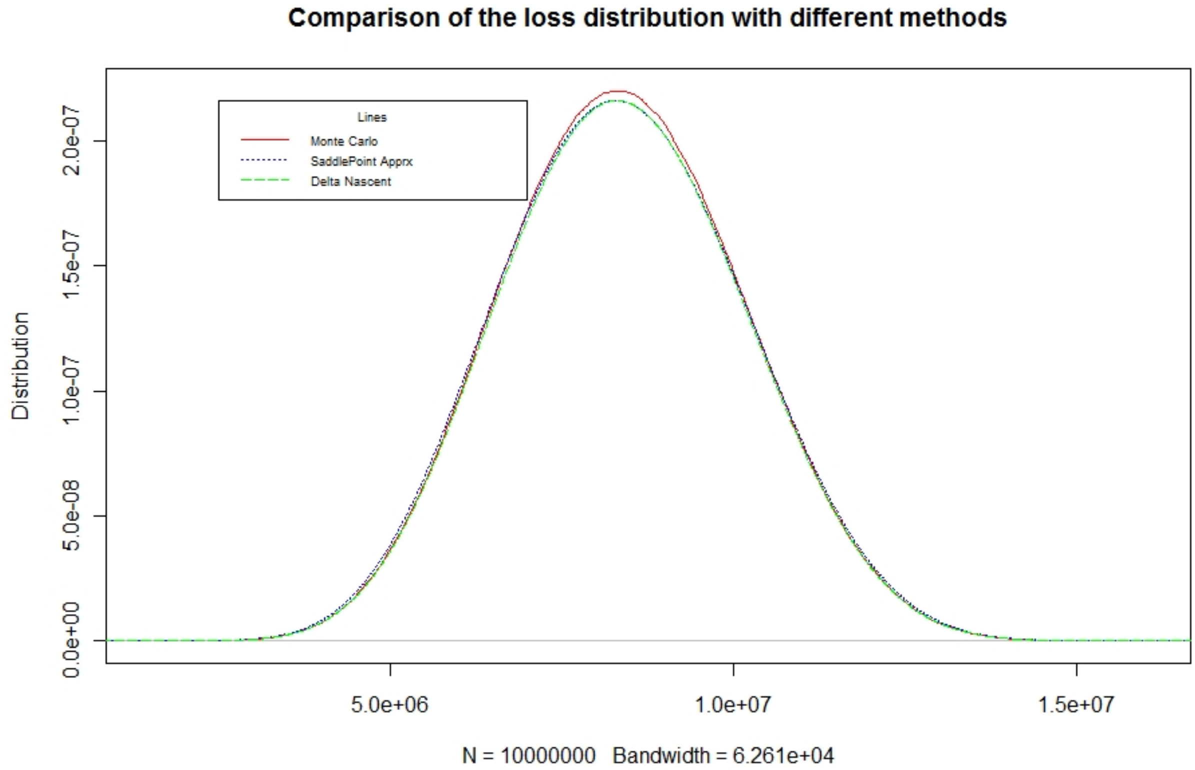

Figure 1: Two segments with two variables. Ex (4.1.1)

4.2 Results

Here we present the results for different methods. We fix the number of simulation 10,000,000 for Monte Carlo method and we use the same discretization for the Nascent Delta method, the direct saddlepoint method and the analytical solution. We discretize the loss interval into 2,000 pieces. The VaR is calculated at 99.9% quantile. The parameter a for the Nascent Delta is set to the same as the step length for discretizing the

4.2.1 Results from different methods for Example (4.1.1)

Here, since we do not have analytical solution in this case, we use the Monte Carlo simulations as benchmark.

First, for Expected Loss

	Expected Loss	"*Error*"	*RelativeError(percentage)*
MonteCarlo	8388314		
Saddlepoint	8378436	−9878	0.118%
NascentDelta	8387252	−1062	0.013%

And for Value at Risk

	Value at Risk	"*Error*"	*RelativeError*
MonteCarlo	13552067		
Saddlepoint	13607536	55469.3	0.41%
NascentDelta	13553536	1469.3	0.011%

The time consumed:

	Time(seconds)
MonteCarlo	27.28
Saddlepoint	11.06
NascentDelta	45.47

4.2.2 Results from different methods for Example (4.1.2)

In this example, we are able to obtain the analytical solution. So we will use it as the benchmark.
First, for Expected Loss

	Expected Loss	*Error*	*RelativeError(percentage)*
AnalyticalSol	435295.1		
MonteCarlo	435310.7	15.6	0.0036%
Saddlepoint	435295.1	5.5×10^{-08}	1.2×10^{-11}%
NascentDelta	435294.9	−0.16	3.8×10^{-05}%

And for Value at Risk

	Value at Risk	*Error*	*RelativeError(percentage)*
AnalyticalSol	907757.5		
MonteCarlo	907725	−32.5	0.0036%
Saddlepoint	907757.5	0	0%
NascentDelta	907757.5	0	0%

Please note that here the error "0" does not mean that the method is accurate without any error. Here the error is 0 because, we are using the same grid for discretizing the loss. And computing the VaR depends on the this grid. Since the methods have very high levels of accuracy, they found the same point to be the VaR (99.9% quantile).

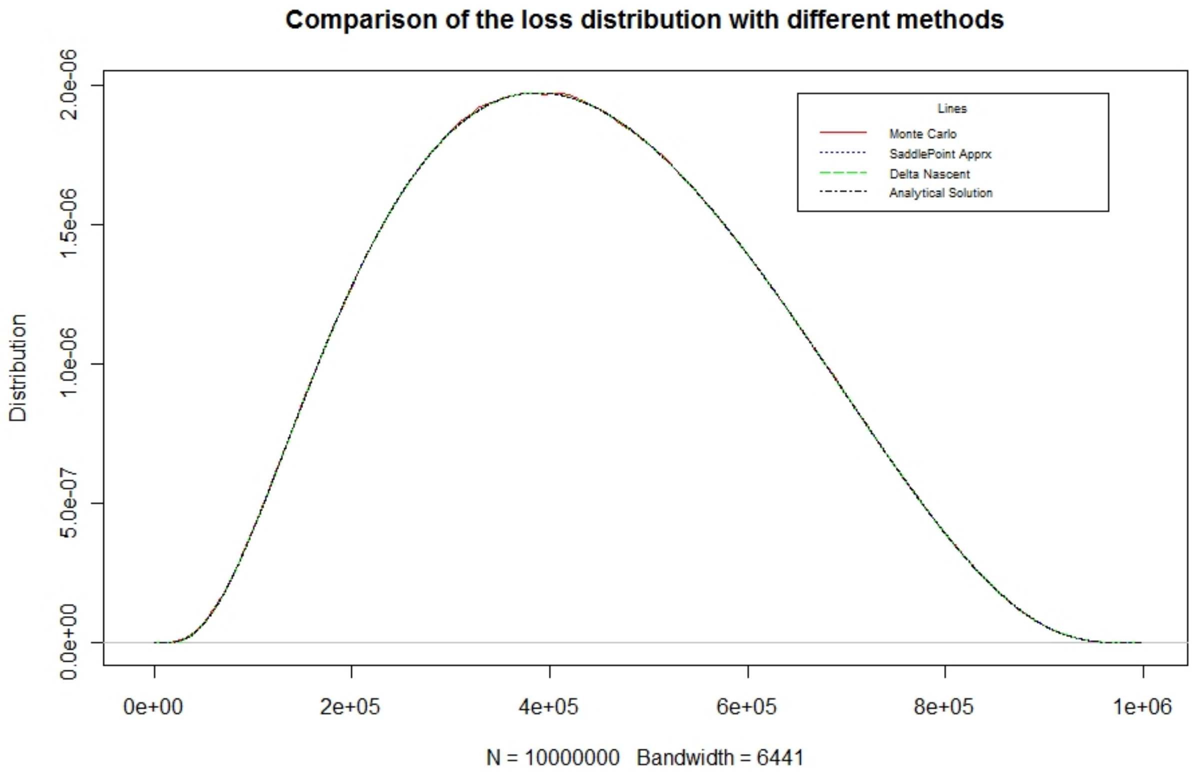

Figure 2: One segment with two variables. Ex (4.1.2)

4.3 Conclusion

As we could see from the examples, the Nascent Delta Method has very high level of accuracy, but it is very time-consuming and one may need extra effort and computation to determine a good parameter a for the method. The Monte Carlo Simulations method is the standard way to calculate the loss distribution, but in order to achieve high level of accuracy, one needs a very large number of simulations, which results in long-time computing and low efficiency. The direct Saddlepoint method has the advantage that it is much faster than the other two methods and the error is tolerable. In the fast-paced financial world, one second ahead of the counterparty may save the company a lot of money. So the direct saddlepoint method should be considered as a very good complementary method for the Monte Carlo Simulations.

Appendices

A Calculating the saddlepoint using iterative method

To compute the saddlepoint which satisfies these conditions in (20), i.e.:

$$\begin{cases} -(X_y - M)^{Tr} \cdot \Sigma^{-1} + \lambda \, {f'_X}^{Tr} [X_y] \ = 0 \\ y - f\,[X_y] = 0 \end{cases} \tag{30}$$

Rearrange the first equation and make Taylor expansion at X_t we have

$$\begin{cases} -\Sigma \, f'_X \,[X_{\mathrm{y}}] \ \ \lambda = M - X_y \\ y - f\,[X_t] - {f'_X}^{Tr} [X_t] \, (X_y - X_t) = 0 \end{cases} \tag{31}$$

Make the iteration scheme as:

$$\begin{cases} (X_{t+1} - X_t) - \Sigma \, f'_X \,[X_t] \ \ \lambda = M - X_t \\ {f'_X}^{Tr} [X_t] \, (X_{t+1} - X_t) = y - f\,[X_t] \end{cases} \tag{32}$$

Make the matrix form

$$\begin{pmatrix} I & -\Sigma \cdot \ f'_X \,[X_t] \\ {f'_X}^{Tr} [X_t] & 0 \end{pmatrix} \cdot \begin{pmatrix} X_{t+1} - X_t \\ \lambda \end{pmatrix} = \begin{pmatrix} M - X_t \\ y - f\,[X_t] \end{pmatrix} \tag{33}$$

Take the inverse matrix,

$$\begin{pmatrix} I & -\Sigma \, f'_X \,[X_t] \\ {f'_X}^{Tr} [X_t] & 0 \end{pmatrix}^{-1} = \frac{1}{\Delta_t} \begin{pmatrix} \Delta_t \mathrm{I}_d - \Sigma \, f'_X \,[X_t] \, {f'_X}^{Tr} [X_t] & \Sigma \, f'_X \,[X_t] \\ -{f'_X}^{Tr} [X_t] & 1 \end{pmatrix} \tag{34}$$

where $\Delta_t = {f'_X}^{Tr} [X_t] \cdot \Sigma \cdot \ f'_X \,[X_t]$

So the explicit solution is:

$$\begin{pmatrix} X_{t+1} - X_t \\ \lambda \end{pmatrix} = \frac{1}{\Delta_t} \begin{pmatrix} \Delta_t \mathrm{I}_d - \Sigma f'_X \,[X_t] \, {f'_X}^{Tr} [X_t] & \Sigma f'_X \,[X_t] \\ -{f'_X}^{Tr} [X_t] & 1 \end{pmatrix} \begin{pmatrix} (M - X_t) \\ y - f\,[X_t] \end{pmatrix} \tag{35}$$

The iteration we will use in numerical computation is:

$$X_{t+1} = \mathrm{M} + \Big(- (\Sigma \, f'_X \,[X_t]) \, {f'_X}^{Tr} [X_t] \, (M - X_t) + (\Sigma \, f'_X \,[X_t]) \times (y - f\,[X_t]) \Big) \frac{1}{\Delta_t} \tag{36}$$

where $\Delta_t = {f'_X}^{Tr} [X_t] \cdot \Sigma \cdot \ f'_X \,[X_t]$

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