Forecasting and planning in chaos

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Abstract

This paper presents a new method for building strategic plans for financial institutions, in a crisis period. The forecasting technique integrates generalized forecasting on multi-equation models, macroeconomic scenarios given by analysis out-of-model and the use of the optimization as an analysis tool. This method output provides the forecast of the internal flows for the institution, using econometric methods, accounting dynamics and optimization criteria in order to obtain the projection of the best decisions showing them as accounting states along the planning horizon. All the statistical errors are integrated in the probability distribution of losses.

Key words: Forecasting, Planning, Optimization, Macroeconomic models

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1 Introduction

Recent crises brought several significant changes in the financial sector and in the way that it is regulated, analyzed, planned and managed, mainly in risk related practices and the integration of the different “silos” in this matter with the rest of financial business components. Regulation is increasingly calling for the utilization of forecasting techniques to evaluate not only capital adequacy, but also the ability to absorb the impact of eventual adverse economic turns or adverse market situations.

Regulation tightening and complexity increase has been a worldwide phenomenon. It is especially relevant in Europe and particularly in countries such as Spain, where the banking sector has been completely restructured, since it received abundant funding resources in order to continue operating under international standards. New stress testing requirement and the development of Basel III regulations enacted, incorporated a sophisticated methodology for the assessment and weighing of risks on top of the previous one, with stronger capital requirements. These were the most severe regulatory measures in Europe. Half the entities operating in 2008 in Spain no longer exist. New banks have been created, including a bad bank named SAREB, which manages bad loans portfolios and intends to sell the tremendous stock of houses, backing mortgage loans, owned by the commercial banks and credit institutions.

Also, the financial markets worldwide became more complex, and this complexity leads financial institutions relentlessly to make organizational changes in order to implement sophisticated analytical tools and a consistent framework for the analysis of unexpected market conditions, in order to be able to respond promptly with adequate strategic business plans and measures.

Since early 2008, when the financial crisis started, the situation is anything but stable. Without stability, forecasting techniques face really significant challenges since a traditional statistics approach is not enough to foresee future events or unexpected market behavior. The previous

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1 In Spain we are subject to both Basel III regulations and government requirements, which are stricter on assets and provisions, as established in royal decrees and laws, along with a new orientation towards knowledge of institutional viability, which involves a future projection of the balance sheet, as set out in Act 9/2012, of 14 November, and in Royal Decree 1559/2012, of 15 November, which compiles all regulations in this regard.
2 Sociedad de gestión de activos procedentes de reestructuración bancaria.
decades had been years of economic growth and expansion. This means that we enjoyed a long period of economic stability in which we were able to use what had happened in the past to forecast the most probable future scenario. The future was somewhat defined by what had happened in the past. Since the financial crisis proved that this is not always the case, a different approach and tools are required to successfully analyze and manage uncertainty. Instruments and tools are required that allow for the combination of expert judgments and views with multiple and changing macroeconomic models, especially to allow efficient decisions, as this has become more critical than ever. The static approach is less important these days, as strategists need a more dynamic analysis of all the business flows and their changes when confronted with fluid macroeconomic environments.

As the Neoclassic School states, every actor tends to seek his or her optimal position in the market. In a perfect market, the general equilibrium would set the prices and amount of money. Prices in equilibrium are fixed near the marginal cost. But the difficulties involved in making forecasts after sudden changes, the lack of information, the decisions of the monetary and regulatory authorities, industry concentration and the dynamics of different products introduce such a degree of noise and complexity that different decisions can be taken, all of which are feasible. Each of these decisions offers different outcomes in terms of value.

In this context, with the instability that rules the economy, it is logical to think that actors are looking for what is the best for them - their optimal position. So it is time for optimization. Optimization emerges as a relevant and crucial element to be considered in the planning and forecasting exercises not only required by the regulators but also as an essential part of sound and intelligent business strategy. It is now essential to plan different strategies to respond to changing macroeconomic conditions and measure the impact in microeconomic models (i.e. risk models, financial models, demand models, etc.), and to simulate scenarios, to introduce out-of-model macroeconomic scenarios, to project balances and P&L (Profit and Loss) statements, in order to determine whether business objectives are achieved.

In this new scenario, this paper proposes an alternative tool to traditional prediction that attempts to incorporate the dynamic consideration of regulation, the judgments of experts, and changes in the macroeconomic environment. The objective is to provide financial institutions a tool to determine the best way to comply with new regulations and other restrictions by using
optimization techniques. Alternatively the system is able to show in feasibility issues as a consequence of contradictory restrictions.

The rest of the paper will proceed as follows: in Section 2, the methodology will be explained in detailed; Section 3 presents the main results of the empirical application; and, finally, Section 4 presents the main conclusions.

2 Methodology

Arrow (1950) gave mathematical consistency to the idea that the integration of individual interests brings about the best application of resources in an economy, and this was the start of the development of general equilibrium theory, laying the groundwork for neoclassical economic theory. At practically the same time, Markowitz (1952) and Tobin (1958), as well as other authors, developed the idea of the operational translation of Arrow's seminal idea to the efficient allocation of assets by agents on financial markets.

Asset allocation is the investment strategy which best balances risk and return of a portfolio, as it provides the combination with greatest profitability for a specific risk. The term was originally applied to equity investments in Markowitz's (1952) portfolio theory, since then, despite the significant constraints of information, market performance and volatility, asset allocation has been a fundamental part of all investment theories.

Pyle (1971) extended the concept of portfolio optimization to all of a bank's assets and liabilities, with liabilities being understood as short selling. In addition to the work of Pierce (1972) and Cohen (1972), other authors developed an economic theory of banking and the criteria for its practical application. Balternsperger's article (1980) includes an analysis of the different approaches to the subject.

We are looking at the possibility of focussing the bank's balance sheet by pondering the best combination of risk and profitability on the asset side and deciding what the best position of liabilities and capital might be. This should be the summary of the financial theory transferred to
the balance sheet of a financial entity. In a totally non-regulated, market-driven economy, the decision concerning the balance sheet belongs to the entities and its evaluation belongs to the market. The optimization would be a result of the best possible combinations of business rules and constraints without restrictions and the market and analysts would make the final evaluation. But the present situation is that capital and liquidity are subject to major restrictions imposed by the regulators. The optimization process must take into account those restrictions in order to find the best strategy. This concept, for both assets and liabilities, has been used for the decision-making methodology developed in this work, incorporating regulatory restrictions (compared to spontaneous decisions on the market) and improving risk assessment through models with complex scenarios.

The following elements are taken into account in this new approach: first, we defined the restrictions in the form of compliance with capital requirements and liquidity. Second, we estimated a model of macroeconomic forecasting and future projection showing the viability of the business plan under the macroeconomic forecast, given the portfolio of the company and a number of specific scenarios. Finally by means of an optimization function subject to these constraints and this macroeconomic scenario we predict the future business plan performance of an entity for a given risk. The latter resembles the regulatory stress test, but is unlike the static analysis and balance sheet, as possible future events may affect the assets and liabilities is based on complex modeling which aims to project the future balance sheet.

Figure 1 summarizes the methodology for decision-making described below.
In the next section, we describe the method in detail with all the inputs required to complete it. We focus on three main components. First we describe the function to be optimized and the main restrictions. Then we describe the macroeconomic and microeconomic models. Finally, we describe the tool outputs.

2.1 Problem to be optimized

Formally, optimization is the selection of a best element (with regard to certain criteria) from a set of available alternatives.

The optimization problem proposed in this paper was designed to find (local) solutions of mathematical optimization problems of the form:

\[
\begin{align*}
\min_{x \in \mathbb{R}^n} & \quad f(x) \\
\text{s.t.} & \quad g_L \leq g(x) \leq g_U \\
& \quad x_L \leq x \leq x_U
\end{align*}
\]

Eq 1
where $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ is the objective function, and $g(x) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ are the constraint functions. The vectors $g_L$ and $g_U$ denote the lower and upper bounds on the constraints, and the vectors $x_L$ and $x_U$ are the bounds on the variables $x$. The functions $f(x)$ and $g(x)$ can be nonlinear and nonconvex, but should be twice continuously differentiable. Note that equality constraints can be formulated in the above formulation by setting the corresponding components of $g_L$ and $g_U$ to the same value.

We resolve this optimization by means of the Internal Point method\(^3\). Using this optimizer, it is possible to include non-linear restrictions such as economic capital, which is described below.

### 2.1.1 Function to be optimized

Several criteria can be assumed for the function to be optimized. Our proposal is maximizes the value of the entity, which will be obtained as the present value of the cash-flow forecasted by the model, described in the equation below:

$$
\max Z = \sum_{t=1}^{n_{per}} (1 + r)^{-t} CF_t + (1 + r)^{-n_{per}-1} K
$$

Eq 2

Where $r$ is the interest rate, $CF_t$ is the Cash Flow of the period $t$, $K$ is the Net worth Total assets less liabilities and $n_{per}$ is the horizon periods. The discount rate can be assimilated to the PER (Price to Earnings Ratio) of a similar entity. The last term in the equation is added to prevent the model from ignoring the future beyond the last horizon period. Another option would be to add the present value of the perpetual income\(^4\) of the mean of the cash flows.

Moreover, we can define two alternative criteria as the objective function. The first one is given below:

$$
\max Z = \sum_{t=1}^{n_{per}} (1 + r)^{-t} CF + (1 + r)^{-n_{per}-1} \delta
$$

Eq 3

\(^3\) Interior point method: this method is based on the fact that the progress of the solutions occurs inside the feasible region and not through the vertices of the region, as in the Simplex method.

\(^4\) We can define the perpetual income as the set of equal payments made at equal intervals, tending to infinity.
Where \( CF \) is Results plus Depreciation and \( \delta \) is the perpetual income.

The other form is the following:

\[
\max Z = \sum_{t=1}^{nper} (1 + r)^{-t} \frac{CF}{Sh} + (1 + r)^{-nper-1} \delta
\]

Eq 4

Where \( Sh \) is the Shareholders’ equity or Common Equity.

2.1.2 The variables

As far as possible, the variables to be included have been associated with accounts, business flows and parameters of common use in financial analysis and planning. Except for a few general parameters, all the variables refer to a point in time. This means they are the period-by-period part of a series. In general, business flows can be associated with stocks and transactions with flows.

Some variables have a dynamic link between them: stocks and transactions. The value of an account like a loan portfolio in some period is equal to its value in the previous period, plus the new accepted, less the amortization, less the write-off, less the early cancelled, less the portfolio sold.

Other variables are designed to be exogenous; this means that they get their value from outside the program, though you can change them among the different scenarios tested and obtain the outcome of the new scenario in a what-if way. The variables could have been considered as exogenous, this is they are not modeled: fixed investments as “Premises and other assets”, different Tiers of Capital, “Capital”, the trading book “Securities for sale” and transactions like directly selling part of a loan portfolio. However, these values will be ready to be obtained using a suitable model.

Finally, other variables are obtained by means of a model, using several formulas which link them to the macroeconomic variables or to others from the microeconomic model (see Section 2.3 and 2.4). The role of risk drivers will be assigned to macroeconomic variables, industry indexes and residual errors of microeconomic models, Probability of Default (PD), Loss Given Default (LGD), Exposure at default (EAD) and loan demand if modeled. We need to introduce values for parameters which are in more detail in the table in Annex 3.
In the same way, the balance-sheet flows can be calculated using several methods. For instance, we can generate them analytically. This is the case of Depreciation, Past Due Loans to write off, etc. Alternatively, we can create restrictions to modify the optimizer output. This is the case of New loans, portfolio for sale or sales of the impaired portfolio. (See annex 3).

It is necessary to introduce a real balance sheet and a P&L accounts as an input. The method uses the information in these instruments to project several periods forward.

### 2.2 The constraints

In this section we describe the main restrictions included in the optimization problem. The restrictions will be divided in four groups: account restrictions, regulatory restrictions, economic capital and business restrictions.

#### 2.2.1 Account restrictions

Some examples of the main restrictions of balance sheet, expressing the movement of the balance are defined in this section.

The dynamics of the balance-sheet loans accounts are defined in the equation below:

\[
StL_t = StL_{t-1} + NewStL_t - AmrStL_t - DelCredStL_t - PrePStL_t + RevStL_t \quad \text{Eq 5}
\]

Where \( StL_t \) is the stock account outstanding on portfolio \( L \), \( NewStL_t \) are the new loans of portfolio \( L \) at period \( t \), \( AmrStL_t \) is the loan amortization, \( DelCredStL_t \) are the delinquent credits \( PrePStL_t \) are the prepayments and \( RevStL_t \) are the recovered loans from past due ones. We must define a different calculation from exposure for each kind of credit (loans, commercial credits, credit cards, guarantees).

In respect to the past due loans we define:

\[
PStL_t = PStL_{t-1} + DelCredStL_t - PDL2WOSTL_t - RevStL_t \quad \text{Eq 6}
\]

Where \( PStL_t \) is the stock of past due loans of \( L \) at a period \( t \) and \( PDL2WOSTL_t \) is the past due loans to write off of the same portfolio \( L \) at the same period.
The write-off flow is calculated using the Probability of Default at each period, the Losses Given Default and a time’s structure which contains the behavior of the impaired loans. And the stock of provisions for credit losses is calculated according to the regulatory standards of the country.

2.2.2 Regulatory restrictions

The regulatory restrictions according to Basel II & III, defined in the method are the capital restrictions, leverage restrictions and liquidity restrictions.

Capital restrictions are defined as:

$$RC\% \leq \frac{CECapital - RM\% \cdot MarketRisk - RO\% \cdot OperationalRisk}{\sum_i (RWA_i \cdot \text{CExposure}_i)}$$  \hspace{1cm} \text{Eq 7}

Where $CECapital$ is the Common Equity Tier 1, or Tier 1 Capital or Total Capital (Tier 1 Capital plus Tier 2 Capital), $RWA_i$ are the risk weight assets, $\text{CExposure}_i$ is the exposure of credit portfolio “i”, $RM\%$ is the minimum coefficient of Market Risk, $RO\%$ is the minimum coefficient of Operational Risk and $RC\%$ are the coefficients of Capital Restrictions. They change during all periods according to Basel III restrictions.

The leverage restrictions are defined as:

$$Lev\% \leq \frac{Tier1Capital}{(\sum_i RL\% \cdot \text{CExposure}_i)}$$  \hspace{1cm} \text{Eq 8}

Where $\text{CExposure}_i$ is the exposure of the credit portfolio “i”, $RL\%$ is the coefficient of Leverage of the credit portfolio “i” and $Lev\%$ is the coefficient of Leverage Restriction. Its maximum value is 3%. The objectives of this coefficient are to limit the accumulation of leverage in the banking sector by helping to prevent destabilizing deleveraging processes that may harm the financial system and the economy. They also aim to reinforce capital requirements based on risk, with a simple non-risk based complementary function.

According the Basel III, we can include two liquidity restrictions, the Liquidity Coverage Ratio and the Net Stable Funding Ratio.
The Liquidity Coverage Ratio aims to ensure that an institution maintains an adequate level of available liquid assets to cover the net balance of inflows and outflows in a stress situation for a period of 30 days. The ratio is defined in the equation below:

\[
\frac{\text{Stock of Liquid Assets}}{Outflows - \text{Min(Inflows}; 75\% \times (Outflows)} \geq 100\%
\]

Where \(Outflows\) are the deposits and \(Inflows\) are the contractual inflows from outstanding exposures that are fully performing and for which the bank has no reason to expect a default within the 30-day time horizon.

The Net Stable Funding Ratio is defined as the portion of those types and amounts of equity and liability financing expected to be reliable sources of funds over a one-year time horizon under conditions of extended stress. Required funding is a function of the liquidity characteristics of various types of assets held, Off-Balance Sheet contingent exposures incurred and/or the activities pursued by the institution. So, Net Stable Funding Ratio is defined as:

\[
\frac{\text{Available Amount of Stable Funding}}{\text{Required amount of Stable Funding}} \geq 100\%
\]

Where \(\text{Available Amount of Stable Funding}\) is the Total Capital plus some kind of deposits and \(\text{Required amount of Stable Funding} = \sum_l (\text{Exposure}_l \times \text{RSF Factor}_l)\). RSF Factor (Required amount of Stable Funding) is a table of factors which depends on the type of exposure (for example a 65% for mortgages).

### 2.2.3 Economic Capital

The present method includes equations that relate any investments set with the Economic Capital needed to guarantee solvency at some level of risk appetite.

The Economic Capital restriction is usually applied in a straightforward way: given a set of investments, the component risk models and the risk appetite, the Capital needed has to be calculated. In our case, Capital can be either a variable or a constant. Starting with the usual definitions of risk components and definition of losses, we have developed a method for calculating the integral of the resulting losses.
The literature contains several methods for calculating the integral of the losses. In some cases, they use Montecarlo systems, which require a considerable response time, as well as producing discontinuous distributions. These are not appropriate systems for use in optimization processes, which require rapid responses and functions that can be differentiated in first and, sometimes, second orders.

To obtain responses in a reasonable time frame and with continuous distribution, the literature indicates the use of several simplified methods, which have sacrificed exactness in the concentration/diversification effect. Simplification to a single underlying factor, the method applied in the calculation of capital, as per Basel II, is clearly insufficient for calculating the economic capital of economies only partially correlated in their evolution over time. Multifactorial computation of García Céspedes (2005) ignores, in other cases, the effect of residual errors both in the models of the factors themselves and in the models that explain the behavior of the portfolios vis-à-vis those factors.

Our proposal, which we have called “Cul d’Olla Integral” is an alternative to the Montecarlo method of integration applied on a contour integral which joins losses functions with the probability of co-variants that correlates with them.

The integral to be solved aims to compute the probability of such a level of losses that exceeds the capital available; as the PDF (probability Distribution Function) is multidimensional, it is needed to integrate probabilities along all the cases with each level of losses, integrating afterwards from this level of losses to the infinity in order to obtain the CDF (Cumulative Distribution Function):

$$\varepsilon = \int_y^\infty \int_{X|X|\geq y} \omega(X, \bar{X}, \Sigma_X) \, dX \, dy$$  \hspace{1cm} \text{Eq 11}$$

Where X is a matrix including the macroeconomic variables and the residual errors of the microeconomic models. And \(\bar{X}\) represents the Expected value of macroeconomic variables and residuals and \(\Sigma_X\) is the Covariance matrix of macroeconomic variables and residuals. The

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5 Cul d’Olla Integral belongs to the the system called Risk Dynamics into the future (RDF), built by AIS. RDF allows for stress testing of credit portfolios, for different macroeconomic scenarios, using the usual measurements of PD, LGD and economic capital.
parameter $l_i$ is the investment volume in portfolio i and $l_i^a[X]$ the portfolio losses i given values of macro and microeconomic variables $X$. $K$ is the capital available and $\varepsilon$ the probability of bankruptcy (measure if risk appetite).

The $\omega[X, \bar{X}, \Sigma_X]$ is the joint distribution $X$. The Mulinormal distribution is express as:

$$
\frac{\exp{-\frac{1}{2}(X-\bar{X})^\top \Sigma_X^{-1}(X-\bar{X})}}{(2\pi)^{n/2}|\Sigma_X|^{1/2}}
$$

In Annex 2, we provide the technical explanation of a system that can respond much faster than the current methods and, of course, faster than the Montecarlo method.

The new approach has been developed, using FT (Fourier Transform - Integral) as a tool for the integration of the Cul d’Olla Integral.

### 2.2.4 Business restrictions

Finally, business restrictions can also be included in the optimization problem we present here. This mean that each flow, ratio or stock at each period can be limited to its higher or lower current value and to its relative increase or decrease margin. The follow figures describe this kind of restrictions:

**Figure 1: Maximum or Minimum**

**Figure 2: Increase or Decrease Margin**

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Restriction  Restricted flow, ratio or stock  Solution space

### 2.3 Macroeconomic Models

Classical macroeconomic forecasts typically ignore expert judgment and assume Ceteris Paribus in the traditional theory. In this paper, we show how overcome this. We explicitly introduced two elements into the classical estimation framework: a subjective guess on the variable to be forecast and a joint multivariate distribution. This conditioned joint distribution contains the probability of
all possible results of a given scenario, without assuming Ceteris Paribus as in traditional economics theory.

Thus this method overcomes some limitations of the traditional methods. The following demonstrates how Multivariate distribution is derived from a VAR model and how macroeconomic scenarios conditional on the judgments are calculated.

2.3.1 VAR Model

Vector autoregressive (VAR) models have been widely used in empirical studies of macroeconomic issues since they were launched for such purposes by Sims (1980). They are now widely used in all kinds of empirical macroeconomic studies, from relatively non-theoretical exercises such as data description and forecasting, to tests of fully specified economic models.

First we estimate an un-restricted VAR model and then apply a Cholesky decomposition to identify the structural model. The VAR with p lags can be defined as follows:

\[ Y_t = c + \Gamma_1 Y_{t-1} + \Gamma_2 Y_{t-2} + ... + \Gamma_p Y_{t-p} + u_t \]  

Eq 12

Where \( c \) is a column vector of constants, \( \Gamma_p \) is a matrix of coefficients, \( Y_t \) is a vector of endogenous variables, and \( u_t \) is the error term. The error term is independent and does not correlate across time.

We can specify the reduced model as a structural model, where the endogenous variable depends on the contemporary values of other endogenous variables (they are contemporary correlated) and on its past values. A structural VAR model with p lags, denoted by SVAR(p), is: (see Annex 1.1)

\[ \text{The Cholesky decomposition establishes a lower triangular scheme. The arrangement of variables in the lower triangular scheme is determined under an economic perspective on the contemporary causal relations. Once the arrangement is established, coefficients can be estimated by means of Cholesky decomposition.} \]
Where \( C_0 \) is a column vector of constants, \( \Phi_p \) is a matrix of coefficients with \( \Phi_0 \neq \text{Identity} \) and invertible, \( Y_t \) is a vector of endogenous variables, and \( \varepsilon_i \) is the error term. The error term is independent and does not correlate across time. In other words, structural shocks are uncorrelated.

### 2.3.2 Forecasting with VAR model

To perform predictions of variables included in VAR model, is transcribed as a Moving Average Process MA(q) model because, in this format, the vector of variables depends only on the list of the previous model's error vector.

We will denote:

\[
Y_t(l) = \text{macroeconomic variables vector in the } t+l \text{ predicted moment.}
\]

If \( \Phi(B)Y_t = \varepsilon_t + c_0 \) is the structural VAR model with \( p \) lags then we obtain the VAR reduced model of \( p \) order:

\[
\Phi_0^{-1}\Phi(B)Y_t = \Phi_0^{-1}\varepsilon_t + \Phi_0^{-1}c_0
\]

\[
\Gamma(B)Y_t' = E_t' + c
\]

With \( E_t = \Phi_0^{-1}\varepsilon_t, \ c = \Phi_0^{-1}c_0 \)

How the MA reduced model is obtained from VAR reduced model, and afterwards, how the MA structural model is obtained is explained in more detail in Annex 1.2 and 1.3.

The MA reduced form is described in the equation below:

\[
Y_t = \Psi(B)(E_t + c) = \left( \sum_{k=0}^{\infty} \Psi_k B^k \right)(E_t + c) = \left( \sum_{k=0}^{\infty} \Psi_k (E_{t-K} + c) \right) = \left( \sum_{k=-\infty}^{0} \Psi_{-K} (E_{t+k} + c) \right) \tag{Eq 14}
\]

Where \( \Psi_k = \Phi_0^{-1} \left( \sum_{j=1}^{K} \Phi_j \Psi_{k-j} \right) \)
The MA representation can be used to predict macroeconomic variable values. Subsequent moments predicted to the original predicted moment $t$ from the macroeconomic variables vector, $Y_t(l) = Y_{t+l}$, are given by the following expression:

$$Y_t(l) = \left\{ \sum_{k=0}^{\infty} \Psi_k (E_{t+l-k} + c) \right\} = \left\{ \sum_{k=-\infty}^{0} \Psi_{l-k} (E_{t+k} + c) \right\} + \left\{ \sum_{k=1}^{l} \Psi_{l-k} (E_{t+k} + c) \right\}$$

Eq 15

Where there are errors from $t$ time backward in the first summation, that is, past ones, and future errors in the second summation.

Next, the structural model’s equivalent equation of the previous Eq. 15 is shown.

$$Y_t(l) = \left\{ \sum_{k=-\infty}^{0} \Xi_{l-k} (\varepsilon_{t+k} + c_0) \right\} + \left\{ \sum_{k=1}^{l} \Xi_{l-k} (\varepsilon_{t+k} + c_0) \right\}$$

Eq 16

In this way, it is possible to calculate the expected value $Y_t(l)$:

$$E(Y_t(l)) = \left\{ \sum_{k=-\infty}^{0} \Psi_{l-k} E(E_{t+k} + c) \right\} + \left\{ \sum_{k=1}^{l} \Psi_{l-k} E(c) \right\}$$

Eq 17

$$= \left\{ \sum_{k=-\infty}^{0} \Xi_{l-k} E(\varepsilon_{t+k} + c) \right\} + \left\{ \sum_{k=1}^{l} \Xi_{l-k} E(c) \right\}$$

Since it is supposed that the future error’s expected value is void as it is random, and past errors have a certain expected value. This way of calculating expected value is theoretical and it's impossible to put into practice, as it is an infinite values summation. So, using VAR reduced model, it is possible to also calculate the expected value:

Where: $E(Y_t(j)) = Y_{t+j}$ for $j \leq 0$

In consequence, the random part of the relation can be written as:
\[
Y_t(l) - \mathbb{E}(Y_t(l)) = \sum_{j=1}^{l} \Psi_{l-j} \xi_{t+j} = \sum_{j=1}^{l} \xi_{l-j} \xi_{t+j}
\]  
Eq 18

For more detail, see Annex 1.4

The covariance matrix of a future moment’s prediction is:

\[
\Sigma_{(l,j)} = \sum_{j=1}^{l} \Psi_{l-j} \Phi_0 \Sigma_\varepsilon \Phi_0^T \Psi_{l-j} = \sum_{j=1}^{l} \xi_{l-j} \Sigma_\varepsilon \xi_{l-j}^T
\]  
Eq 19

Therefore, time distribution of macroeconomic variables vector is:

\[
Y_t(l) \approx \Omega_{\mathbb{P}} \left( \mathbb{E}(Y_t(l)); \mathbb{E}(Y_t(l)), \Sigma_{(l,j)} \right)
\]  
Eq 20

Equation 19 is a Unique Multivariate Distribution. This is an innovative method to determine scenarios through a unique and joint distribution. By this means, we obtain an expected variables vector along the complete horizon as its corresponding covariance matrix, integrating all factors.

### 2.3.3 Macroeconomic Scenarios

A scenario is a possible situation in the space of situations defined by the possible values of the underlying variables in the model. In our setting, a scenario is a possible case in the space of economic situations, defined by the values, at a moment in time, for all or part of the macroeconomic variables that support the model.

To define a macroeconomic scenario we have to define the macroeconomic variables vector to \( l \) periods, in two parts:

\[
Y_t(l) = \begin{pmatrix}
Y_{1,t(l)} \\
Y_{2,t(l)}
\end{pmatrix}
\]
Where \( Y_{1,t,(l)} \) is a vector of \( t \) variables that move freely. \( Y_{2,t,(l)} \) is a vector of \( s \) economic variables that are fixed and later will be used to define scenarios. \( \Sigma_{ij,t,(l)} \) is the submatrix of correlation of vectors \( i, j \) with \( i,j \in \{1,2\} \).

The distribution function for this vector is given by:

\[
\begin{pmatrix}
  Y_{1,t,(l)} \\
  Y_{2,t,(l)}
\end{pmatrix}
= \Omega \begin{pmatrix}
  \begin{pmatrix} Y_{1,t,(l)} \\ Y_{2,t,(l)} \end{pmatrix} \\
  \begin{pmatrix} \bar{Y}_{1,t,(l)} \\ \bar{Y}_{2,t,(l)} \end{pmatrix}
\end{pmatrix},
\Sigma = \begin{pmatrix}
  \Sigma_{11,t,(l)} & \Sigma_{12,t,(l)} \\
  \Sigma_{21,t,(l)} & \Sigma_{22,t,(l)}
\end{pmatrix}
\]

Eq 21

After this we can define a conditional macroeconomic scenario.

After fixing \( Y_2 = Y_2^* \) and

\[
\Sigma_{Y_2^*} = \begin{pmatrix}
  \text{var}(Y_{1,2}^*) & 0 & \cdots & 0 & 0 \\
  0 & \text{var}(Y_{2,2}^*) & \cdots & \vdots & 0 \\
  \vdots & 0 & \ddots & 0 & \vdots \\
  0 & \vdots & \cdots & \text{var}(Y_{S-1,2}^*) & 0 \\
  0 & 0 & \cdots & 0 & \text{var}(Y_{S,2}^*)
\end{pmatrix},
\]

We are defining a scenario. Applying the Schur complement, the resultant marginal distribution, will still be normal with:

\[
\begin{align*}
E\{Y_{1,t,(l)} | Y_{2,t,(l)} = Y_{2,t,(l)}^*\} &= \bar{Y}_{1,t,(l)} + \Sigma_{12,t,(l)} \Sigma_{22,t,(l)}^{-1} (Y_{2,t,(l)}^* - \bar{Y}_{2,t,(l)}) \\
\Sigma\{Y_{1,t,(l)} | Y_{2,t,(l)} = Y_{2,t,(l)}^*\} &= \Sigma_{11,t,(l)} - \Sigma_{12,t,(l)} \Sigma_{22,t,(l)}^{-1} \Sigma_{21,t,(l)}
\end{align*}
\]

Eq 22

Thus we have:
With the Schur complement we obtain the Normal distribution conditioned to the scenario, i.e., the average vector and the covariance matrix conditioned to the fixed values variables that define a scenario. This last method allows us to calculate the conditional scenarios according to expert judgment.

2.4 Microeconomic Models

The first studies that examined the statistical models to predict defaults used discriminant methods (see Altman, 1969; Van Wicklen Stuhr, 1974 and Sinkey, 1975). However, this methodology had serious disadvantages, one of which is the assumption of normally distributed regressors. Because, in general, financial ratios are not normally distributed, since the 1980s maximum-likelihood methods have been used more frequently (see Lennox [1999]). Probit and logit appear to more appropriate models for estimating the probability of default not only for statistical reasons but also because they directly estimate PDs. (see Martin, 1977; Chesser 1974, Lennox , 1999; among others).

Following the literature, we use a logit model to estimate the main measures of risk (PDs and LGDs). The logit model was estimated as a linear regression of the main macroeconomic variables to explain this variable. The equation is defined below:

$$ PD = \frac{1}{1 + \exp(-s_{it})} \tag{Eq 24} $$

Where,

$$ s_{it} = \beta_0 + \beta_1 x_{t-1} + e_{it} \quad \text{With } l = 0, 1, \ldots, t $$
Where $\beta_0$, $\beta_1$, and $\beta_2$ are the parameters associated with the main macro-determinants of PDs found in the vector $X_{t-1}$. It should be noted that the macroeconomic variables may be lagged one or more periods because of the definition of the PDs. All variables are in logarithmic first difference. The macroeconomic variables selected were determined based on the main findings of the literature and the correlation between PD and these variables.

In addition to classical risk measures, a model for loan demand was estimated. Private credit demand was estimated as a linear function of the main macroeconomic determinants, as indicated in the literature (Calza et al, 2001; Gattin-Turkalj et al, 2007). The equation is described below:

$$LOANS_t = \beta_0 + \beta_1 GDP_t + \beta_2 SI_t + \beta_3 LI_t + \epsilon_t$$

Where, LOANS is a measure of private credit growth, GDP is the gross domestic product growth in real terms; SI is the interest rate on short-term and LI is the rate of long-term interest. All variables are expressed in logs.

### 2.5 Outputs

The main outputs of the tool are projections, ratios, different kinds of graphics and information about restrictions (post optimum analysis).

Once the optimum has been found, the method provides a projection of the balance sheet, the statement of Cash Flow and the performances of each portfolio for each future period. The method also shows the value of different kinds of ratios (Basel ratios, global ratios and ratios of portfolios) for each past and future period.

Moreover, it is possible to analyze the restrictions of the solution with the objective of determining where the optimum is positioned and validating what would happen if we were to modify a restriction. Thus, the constraints, whether equalities or inequalities, are controlled by slack and dummy variables. Several analyses can be performed using them.
If we obtain a feasible solution (a valid solution has been reached) we can identify the inactive restrictions and distance to become active. Otherwise, for the active restrictions, we obtain the profit contribution (dual analysis) and the distance to become inactive.

If we obtain an unfeasible solution (no legal solution can be obtained) we can identify the constraints inconsistency.

3 Empirical Application

In this section, we will see the comparison between the business strategy followed by a financial institution and the strategy proposed by the method described. To this end, we have used Strategic Advisor\(^7\), a tool designed by us that uses the method described.

To perform this exercise, we have used the consolidated date of a Spanish credit institution.\(^8\) The main macroeconomic variables included in the model are the following: Gross domestic product (GDP), private consumption, unemployment rate, CPI, housing prices, wage income and interest rates. The main sources were the Bank of Spain, the Ministry of Competitiveness and the National Institute of Statistics. The study was carried out on the periods between the first quarter of 2012 and the third quarter of 2013, as these were the latest periods for which data was available.

The simulation exercise consists of starting with the first real period, the first quarter of 2012, and optimizing the balance sheet, subject to the imposed restrictions, for the next six quarters. The business plan obtained is then compared with the real evolution of the studied institution.

The following restrictions were imposed to perform the optimization:

i. The levels of capital are the same as the real levels for all the remaining periods.
ii. The demand for mortgages is limited.
iii. The liquidity ratio cannot decrease in any period.

---

\(^7\) Strategic Advisor is a web program based on the method describing in this paper
\(^8\) In order to keep anonymous the values presented have been encrypted.
iv. New fixed-term deposits can grow by a maximum of 35% (depending on the type of deposit).

v. The indicated risk appetite is 0.05%.

The objective function used in this trial is the Shareholder value added maximum (See Eq. 4).

3.1 Results

In this section, we present the main results of the exercise, i.e., we will show a comparison between the optimum business plan (based on the criteria imposed) and the real business plan carried out by the studied financial institution.

As can be seen in Figure 2, the optimum strategic plan is more ambitious than that carried out by the institution. As can be deduced from the figures, it was decided to reduce the level of investment while minimizing debt requirement. The optimum business plan, however, increased investment and, therefore, required more funding to finance the investment.

Figure 2 shows that the optimum strategy obtains better results while requiring even less cash flow than the real institution.
This behavior is due to the fact that the optimum plan obtains many sight and short-term deposits, which pay less interest, and invests in more profitable portfolios, considering the imposed risk appetite. In this way, the interest and yields assimilated grow at a greater rate than the interests and charges paid (See figure 5 at Annex 3).

Finally, Figure 4 shows the different capital ratios in the last period (See Eq.7). As we can see, the regulatory ratios from the optimum business plan, because the plan is less conservative, remain below the real ratios of the institution in the same period. Nevertheless, the ratios are well above the minimum ratios required by the regulatory committee.
4 Conclusions

This method is a major step forward in terms of managing and/or supervising an institution. This is largely thanks to its efficacy, robustness and adaptability.

The result of combining the projections of the macroeconomic models with expert opinions and subsequently being able to determine which is the best route to take opens the way to innovative stress-testing techniques.

This study also demonstrates a way of diversifying the portfolio that has been unexplored to date. The ability to include the economic capital restriction in an optimization model was practically unthinkable. Current methods require many simulations to determine a capital value, given a level of investment and risk appetite. In this method, only the risk appetite is given; it is the optimization itself that determines the economic capital and the investment strategy.

In the example provided, we have seen one of its many applications: backtesting. This example explicitly shows the difference between using the method and not using it; the results obtained would have improved considerably. Not only this but they would have been the best possible results given the optimization function chosen and the restrictions imposed. Hence, they are optimum.

Finally, the combination of the different elements that make up this method and the innovation of the optimization as a calculation engine make it an excellent alternative to traditional methods.

References


Annexes

1. Macroeconomic Model

1.1 Equivalence of reduced and structural model

Both representations are equivalent and the relation between equation 1 and equation 2 is given by:

\[ Y_t = \Phi_0^{-1} c_0 + \Phi_0^{-1} \Phi_1 Y_{t-1} + \Phi_0^{-1} \Phi_2 Y_{t-2} + \ldots + \Phi_0^{-1} \Phi_p Y_{t-p} + \Phi_0^{-1} \epsilon_t \]  \hspace{1cm} \text{Eq 26}

If element to element is identified, we have:

\[ c = c_0, \Gamma_1 = \Phi_0^{-1} \Phi_1, \Gamma_2 = \Phi_0^{-1} \Phi_2, \ldots, \Gamma_p = \Phi_0^{-1} \Phi_p, E_t = \Phi_0^{-1} e_t \]

Reduced form errors \( E_t \) are linear combinations of structural errors \( \epsilon_t \) and have a covariance matrix:

\[ \Sigma_E = \text{Cov}(E_t) = E(E_t'(E_t)^T) = \Phi_0^{-1} E(\epsilon_t'(\epsilon_t)^T) (\Phi_0^{-1})^T = \Phi_0^{-1} \text{Cov}(\epsilon_t)(\Phi_0^{-1})^T = \Phi_0^{-1} \Sigma_\epsilon (\Phi_0^{-1})^T \]  \hspace{1cm} \text{Eq 27}

Since \( \Sigma_E \) is a positive-definite matrix (since it is a covariance matrix) then its Cholesky decomposition exists, \( \Sigma_E = PP^T \), with \( P \) lower triangular matrix is the Cholesky triangle.

Our methodology imposes that \( \Phi_0 \) is a lower triangular matrix with 1’s on the diagonal. \( \Phi_0 \) is defined as:

\[ \Phi_0 := DP^{-1}, \text{ Where } D \text{ is a diagonal matrix}. \]

Therefore, we have, \( D = \text{diagonal}(P) \).

On the other hand, it is observed,

\[ \Sigma_E = \Phi_0^{-1} \Sigma_\epsilon (\Phi_0^{-1})^T = PD^{-1} \Sigma_\epsilon (D^{-1})^T P^T = PP^T \]

\[ \Rightarrow D^{-1} \Sigma_\epsilon (D^{-1})^T = Id \Leftrightarrow D = (\Sigma_\epsilon)^{1/2} \]

Then,

\[ D = \text{diagonal}(P) = (\Sigma_\epsilon)^{1/2} \]  \hspace{1cm} \text{Eq 28}

From now on the structural and reduced model will be expressed as follows:

\[ \text{SVAR}(p) \quad \text{VAR}(p) \]
\[ \Phi(B)Y_t = \epsilon_t + c_0 \quad \Gamma(B)Y_t = E_t + c \]

In an analogous way, it is possible to deduce \( \text{VAR}(p) \) from \( \text{SVAR}(p) \):
Multiplying on both sides of the structural form by: $\Phi^{-1}_0$

$\Phi_0^{-1} \Phi(B) Y_t = \Phi_0^{-1} \epsilon_t + \Phi_0^{-1} c_0$

con:

$\Gamma(B) = \Phi_0^{-1} \Phi(B), E_t = \Phi_0^{-1} \epsilon_t, c = \Phi_0^{-1} c_0$

$\left( \Gamma_0 - \Gamma_1 B - \Gamma_2 B^2 - \ldots - \Gamma_p B^p \right) = \Phi_0^{-1} \Phi(B) Y_t = \Phi_0^{-1} \Phi(B) Y_t$

Equivalently

$\Gamma_0 = I_0, \Gamma_1 = \Phi_0^{-1} \Phi_1, \Gamma_2 = \Phi_0^{-1} \Phi_2, \ldots, \Gamma_p = \Phi_0^{-1} \Phi_p$

$E_t = \Phi_0^{-1} \epsilon_t, c = \Phi_0^{-1} c_0$

And the errors covariance matrix is: $\Sigma_E = \Phi_0^{-1} \Sigma_{\epsilon}(\Phi_0^{-1})^T$

1.2 MA reduced model

If $\Phi(B)Y_t = \epsilon_t + c_0$ is the structural VAR model with $p$ lags then we obtain the VAR reduced model of $p$ order:

$\Phi_0^{-1} \Phi(B) Y_t = \Phi_0^{-1} \epsilon_t + \Phi_0^{-1} c_0$

$\Gamma(B)Y_t = E_t + c$

with

$\Gamma(B) = \Phi_0^{-1} \left( \Phi_0 - \Phi_1 B - \Phi_2 B^2 - \ldots - \Phi_p B^p \right) = \Phi_0^{-1} \Phi_0 - \Phi_0^{-1} \Phi_1 B - \Phi_0^{-1} \Phi_2 B^2 - \ldots - \Phi_0^{-1} \Phi_p B^p$

$E_t = \Phi_0^{-1} \epsilon_t, c = \Phi_0^{-1} c_0$

$\Gamma(B)^{-1}$ exists if:

$\det(Id - \Phi_0^{-1} \Phi_1 B - \Phi_0^{-1} \Phi_2 B^2 - \ldots - \Phi_0^{-1} \Phi_p B^p ) \neq 0$ \quad for $|z| \leq 1$

Eq 30

The existence of $\Gamma(B)^{-1}$ is assured because the series used are non-cointegrated.

Now, applying $1 \Gamma(B)^{-1}$ to both sides of the reduced model, we obtain:

$Y_t = \Gamma(B)^{-1} (E_t + c)$

We denote $\psi(B) = \Gamma(B)^{-1}$, that is, $\Gamma(B) \psi(B) = Id$, which delivers:
\[
\left(\Phi_0^{-1}\Phi_0 - \Phi_0^{-1}\Phi_1 B - \ldots - \Phi_0^{-1}\Phi_p B^p \right) \left(\Psi_0 + \Psi_1 B + \Psi_2 B^2 + \ldots + \Psi_p B^p + \ldots \right) = id
\]  
Eq 31

Identifying coefficients, we obtain:

\[
\Psi_0 = id
\]

\[
\Psi_1 = \Phi_0^{-1}(\Phi_1 \Psi_0)
\]

\[
\Psi_2 = \Phi_0^{-1}(\Phi_1 \Psi_1 + \Phi_2 \Psi_0)
\]

\ldots

\[
\Psi_k = \Phi_0^{-1} \left( \sum_{j=1}^{k} \Phi_j \Psi_{k-j} \right)
\]

And finally, we obtain the MA reduced form:

\[
Y_t = \Psi(B)(E_t + c) = \left( \sum_{k=0}^{\infty} \Psi_k B^k \right)(E_t + c) = \left( \sum_{k=0}^{\infty} \Psi_k (E_{t-k} + c) \right) = \left( \sum_{k=-\infty}^{0} \Psi_{-k} (E_{t+k} + c) \right)
\]  
Eq 32

1.3 MA structural model

Departing from MA reduced form:

\[
Y_t = \Gamma(B)^{-1}(E_t + c)
\]

\[
= \Psi(B)(E_t + c)
\]

from which the MA structural model is easily obtained, since: \(E_t = \Phi_0^{-1} \varepsilon_t\)

\[
Y_t = \Psi(B)\Phi_0^{-1} (\varepsilon_t + c_0)
\]

\[
= \Xi(B)(\varepsilon_t + c_0)
\]

where:

\[
\Xi(B) = \sum_{k=0}^{\infty} \Xi_k B^k
\]

\[
= \Psi(B)\Phi_0^{-1}
\]

\[
= \Phi_0^{-1} + \Psi_1 \Phi_0^{-1} + \Psi_2 \Phi_0^{-1} + \ldots
\]

That is:
\[ \Xi_0 = \Phi_0^{-1} \]
\[ \Xi_k = \Psi_k \Phi_0^{-1} \]  
\text{Eq 33}

1.4 Expected value and error covariance projection

\[ Y_t(l) = \Psi_0(E_{t+l} + c) + \Psi_1(E_{t+l-1} + c) + \Psi_2(E_{t+l-2} + c) + \ldots = \left( \sum_{k=0}^{\infty} \Psi_k (E_{t+l-k} + c) \right) \]
\[ = \left( \sum_{k=-\infty}^{l} \Psi_{l-k} (E_{t+k} + c) \right) = \left( \sum_{k=-\infty}^{0} \Psi_{l-k} (E_{t+k} + c) \right) + \left( \sum_{k=1}^{l} \Psi_{l-k} (E_{t+k} + c) \right) \]  
\text{Eq 34}

Next, the structural model equivalent equation of the previous Equation 34 is shown.

\[ Y_t(l) = \left( \sum_{k=-\infty}^{0} \Xi_{l-k} (e_{t+k} + c_0) \right) + \left( \sum_{k=1}^{l} \Xi_{l-k} (e_{t+k} + c_0) \right) \]  
\text{Eq 35}

In this way, it is possible to calculate the expected value \( Y_t(l) \):

\[ E(Y_t(l)) = \left( \sum_{k=-\infty}^{0} \Psi_{l-k} E(e_{t+k} + c) \right) + \left( \sum_{k=1}^{l} \Psi_{l-k} E(c) \right) \]
\[ = \left( \sum_{k=-\infty}^{0} \Xi_{l-k} E(e_{t+k} + c) \right) + \left( \sum_{k=1}^{l} \Xi_{l-k} E(c) \right) \]  
\text{Eq 36}

This way of calculating expected value is theoretical and it is impossible to put into practice, as it is an infinite values summation.

So, using the VAR reduced model, it is possible to also calculate the expected value:

\[ E(Y_t(l)) = E(Y_{t+l}) = E(\Gamma_1 B + \ldots + \Gamma_p B^p Y_{t+l}) + E(E_t + c) = \]
\[ = \Gamma_1 E(Y_{t+l-1}) + \ldots + \Gamma_p E(Y_{t+l-p}) + E(c) = \Gamma_1 E(Y_{t+l-1}) + \ldots + \Gamma_p E(Y_{t+l-p}) + c \]  
\text{Eq 37}

Where: \( E(Y_t(j)) = Y_{t+j} \) for \( j \leq 0 \)

Next, the first future expected values appear:

\[ E(Y_t(1)) = E(Y_{t+1}) = \Gamma_1 Y_t + \Gamma_2 Y_{t-1} + \ldots + \Gamma_p Y_{t-p} + c \]
\[ E(Y_t(2)) = E(Y_{t+2}) = \Gamma_1 E(Y_{t+1}) + \Gamma_2 Y_t + \ldots + \Gamma_p Y_{t+2-p} + c \]
\[ \ldots \]
In consequence, the random part of the relation can be written as:

\[ Y_t(l) - E(Y_t(l)) = \Psi_0 E_{t+1} + \Psi_1 E_{t+2} + ... + \Psi_{t-1} E_{t+1} \]

\[ = \Psi_0 \Phi^{-1} e_{t+1} + \Psi_1 \Phi^{-1} e_{t+2} + ... + \Psi_{t-1} \Phi^{-1} e_{t+1} \]

\[ = \sum_{j=1}^{l} \Psi_{t-j} E_{t+j} = \sum_{j=1}^{l} \Xi_{t-j} E_{t+j} \]

Eq 38

The covariance matrix of a future moment’s prediction is:

\[ \Sigma_{(t,l)} = E\left[ (Y_t(l) - E(Y_t(l))) (Y_t(l) - E(Y_t(l)))^T \right] = E\left[ E_t(l) E_t^T(l) \right] \]

\[ = \sum_{j=1}^{l} \Psi_{t-j} \Phi^{-1} \Xi_e (\Phi_0^{-1})^T \Psi_{t-j} = \sum_{j=1}^{l} \Xi_{t-j} \Xi_e \Xi_{t-j}^T \]

Eq 39

The covariance matrix for all possible random variables used in the modelling process up to \( t + l \) time is calculated as:

\[ \text{cov}(Y_t(l), Y_t(l')) = E\left( \sum_{j=1}^{l} \Xi_{t-j} e_{t+j} \sum_{j=1}^{l'} \Xi_{t-j} e_{t+j} \right) \]

Under suppositions of structural errors:

\[ E(e_{t+j}) = 0 \quad \text{si} \quad j > 0 \]

\[ E(e_{t+j} e_{t+j}) = \begin{cases} 0 & \text{si} \quad j \neq j' \\ \sigma^2 & \text{si} \quad j = j' \end{cases} \]

The covariance matrix can be expressed as:

\[ \Sigma_{(t,l')} = \text{cov}(Y_t(l), Y_t(l')) \]

\[ = E\left( \sum_{j=1}^{l} \sum_{j=1}^{l'} \Xi_{t-j} e_{t+j} \Xi_{t-j} e_{t+j} \right) \]

\[ = \sum_{j=1}^{\min(l,l')} \Xi_{t-j} \Xi_e \Xi_{t-j}^T \]

Eq 40

Then we define:
The integral to be solved is a kind of contour integral, a non-Montecarlo Method was developed: the Cul-d’Olla Integral.

The main lines of this solution are:

- Re-writing the Contour Integral as a Hyper-Surface Integral and using a Dirac Delta Function.
- For each value of losses, identifying the point where the probability of occurrence conditioned to such a value is the maximum one.
- Developing a Taylor function around this point.
- Lowering one degree the integral dimensionality, letting one of the variables be dependent of the rest of them.
- The probability for each loss volume can be obtained by integrating the multi-normal implicit integral.

This procedure is faster than Montecarlo, but it needs to be calculated at each cell of the Losses Distribution if we want to obtain the CDF (Cumulative Distribution Function) at least for each level below the Value at Risk. This sequence of calculations consumes a lot of resources; as each cell needs a new integration, it requires obtaining a new solution to the maximum probability of the set of drivers that produces such a loss.

However, in some processes (e.g. their use as a restriction of capital), where the complete PDF is not required, a new calculation strategy can be set up:
• Re-writing the Contour Integral as a Lebesgue Integral, using a Dirac Delta Function and solving it using Fourier transform.
• Combining it with the Integral from the current Capital to Infinity.
• Changing the integration order moving to the most inner part the last integrand, so the Dirac Delta Function can be converted into a Heaviside Unit Step Function.
• For the level of losses analyzed (i.e. the available Capital), identifying the point where the probability of occurrence conditioned to such a value is the maximum one.
• Developing a Taylor function around this point.
• Thanks to Fourier analysis we can obtain the Characteristic Function of the marginal properties over the line of integration.

This method can be used for the first order Taylor development of the total Losses function.

The integral to be solved intends to compute the probability of a level of losses that exceeds the capital available; as the PDF (probability Distribution Function) is multidimensional, it is needed to integrate probabilities along all the cases with each level of losses, integrating afterwards from this level of losses to the infinity in order to obtain the CDF (Cumulative Distribution Function):

$$\mathcal{E} = \int_{y=K}^{\infty} \int_{X,\bar{X},|X|} \omega[X,\bar{X},\Sigma_X] \, dX \, dy$$

Eq 42

$$X = \text{Macroeconomic Variables and residual error of microeconomic models}$$

$$\bar{X} = \text{Expected Value of Macroeconomic Variables and residuals}$$

$$\Sigma_X = \text{Covariance Matrix of Macroeconomic variables and residuals}$$

$$\omega[X,\bar{X},\Sigma_X] = \text{Join distribution } X. \text{ Multinormal Distribution} = \frac{e^{-\frac{1}{2}(X-M)^T\Sigma^{-1}(X-M)}}{(2\pi)^{\frac{n}{2}}|\Sigma|^{\frac{1}{2}}}$$

$$I_i = \text{Investment volume in portfolio } i$$

$$l_i[X] = \text{losses portfolio } i \text{ given values of macro and microeconomic variables } X$$

$$K = \text{capital available}$$

$$\varepsilon = \text{Probability of bankruptcy. Measure of risk appetite}$$
\[ \delta[z] = \text{Dirac Delta Function} = \delta_a = \frac{1}{a\sqrt{2\pi}} e^{-x^2/2a^2} \text{ as } a \to 0 \]

Using the Dirac Delta function and rearranging,

\[ \varepsilon = \int_{y=K}^{\infty} \int_{X \in \mathbb{R}^n} \delta[L[X] - y] \omega(X, \bar{X}, \Sigma_X) \, dX \, dy \]

Eq 43

Like the Saddelpoint Method, we look for the point with the maximum probability of occurrence (PDF) conditioned to have \( y \) losses, for each value of \( y \). This point can be found using the exponential component of the multi-normal distribution combined with the Lagrange multiplier and the Capital – Losses restriction:

\[ \text{elip}[X] = -\frac{1}{2} (X - \bar{X})^T \Sigma_X^{-1} (X - \bar{X}) - \lambda (L[X] - K) \]

Taking derivatives

\[ \frac{\delta}{\delta X} \text{elip}[X] = -(X - \bar{X})^T \Sigma_X^{-1} - \lambda \frac{\partial}{\partial X} L[X] = -(X - \bar{X})^T \Sigma_X^{-1} - \lambda L'[X] \]

\[ \frac{\delta}{\delta \lambda} \text{elip}[X] = L[X] - K \]

Eq 44

Obtaining the point \( \{X_K, \lambda_K\} \) where the first derivative equals to zero

\[ -(X_k - \bar{X})^T \Sigma_X^{-1} - \lambda_K L'[X_k] = 0 \]

\[ L[X_k] - K = 0 \]

This system can be solved using a multi dimensional Newton-Raphson procedure:
\[
\begin{pmatrix}
X_{K,t+1} \\
\lambda_{K,t+1}
\end{pmatrix} = 
\begin{pmatrix}
X_{K,t} \\
\lambda_{K,t}
\end{pmatrix} - \left(-\Sigma_{X}^{-1} - \lambda_{K,t}L'[X_{K,t}] \right)^{-1} \left(\begin{pmatrix}
L'[X_{K,t}] \\
0
\end{pmatrix} \right) \cdot \left(\begin{pmatrix}
-\Sigma_{X}^{-1}(X_{K,t} - \bar{X}) - \lambda_{K,t}L'[X_{K,t}] \\
L[X_{K,t}] - K
\end{pmatrix} \right)
\]

\[
\begin{pmatrix}
X_{K,0} \\
\lambda_{K,0}
\end{pmatrix} = \begin{pmatrix}
\bar{X} \\
L'[X_{0}](X_{0} - \bar{X})
\end{pmatrix}
\]

Stop at \[\text{Norm}\left[\begin{pmatrix}
X_{K,t+1} \\
\lambda_{K,t+1}
\end{pmatrix} - \begin{pmatrix}
X_{K,t} \\
\lambda_{K,t}
\end{pmatrix}\right] < \varepsilon\]

Eq 45

Taylor development is performed at a point that shows the maximum probability among all the points with a given loss associated.

In the graph, two functions overlap: the PDF function (with an ellipsoid shape) and the losses function (shadowed according to the amount of losses).

An interesting side property of this point is that it is the location for the reverse stress test given the maximum admissible losses.

The integral is done over the probability function using as the integration path the lines shown with the same level of losses. From this border to infinity it integrates the Cumulative Probability Distribution Function.

We can approximate to first order the contour equation and insert it into eq 43:
$$L[X] - y \cong L[X_k] - y + \frac{\partial}{\partial X} L[X_k](X - X_k) = K - y + L'[X_k](X - X_k)$$

$$\varepsilon = \int_{y=K}^{y=\infty} \int_{X \in \mathbb{R}^n} \delta[K - y + L'[X_k](X - X_k)] \omega[X, \bar{X}, \Sigma_X] dX dy$$

Eq 46

Using the theory of distributions, the Cauchy equation can be rearranged to resemble Fourier original formulation and so the $\delta$-function is expressed as:

$$\delta(x - \alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ip(x-\alpha)} dp$$

Eq 47

Using eq. 47 into eq. 46 and rearranging:

$$\varepsilon = \int_{y=K}^{y=\infty} \frac{1}{2\pi} \int_{X \in \mathbb{R}^n} \omega[X, \bar{X}, \Sigma_X] \int_{t \in \mathbb{R}} e^{it(K + L'[X_k](X - X_k) - y)} dt dX dy$$

$$\varepsilon = \int_{y=0}^{y=\infty} \frac{1}{2\pi} \int_{t \in \mathbb{R}} e^{-ity} \int_{X \in \mathbb{R}^n} e^{itL'[X_k](X - X_k)} \omega[X, \bar{X}, \Sigma_X] dX dt dy$$

$$\varepsilon = \int_{y=0}^{y=\infty} \frac{1}{2\pi} \int_{t \in \mathbb{R}} e^{-ity} \int_{X \in \mathbb{R}^n} e^{itL[X_k](X - X_k)} \frac{e^{-\frac{1}{2}(X - \bar{X})^{T} \Sigma_X^{-1} (X - \bar{X})}}{(2\pi)^{n/2} |\Sigma_X|^{1/2}} dX dt dy$$

Eq 48

Integrating the multidimensional part of eq. 48

$$\varepsilon = \int_{y=0}^{y=\infty} \frac{1}{2\pi} \int_{t \in \mathbb{R}} e^{-ity} e^{itL'[X_k](X - X_k) - \frac{1}{2}L'[X_k]^{T} \Sigma_X L'[X_k]} dt dy$$

And integrating the characteristic function,
\[
\varepsilon = \int_{y=0}^{\infty} \frac{e^{-\frac{1}{2} \left( \frac{y-L'[X_k](\bar{X}-X_k))}{\sqrt{L'[X_k]^{T}\Sigma X L'[X_k]} \right)^2}}{\sqrt{2\pi}} \, dy = \frac{1}{2} \text{Erf} \left( \frac{L'[X_k](\bar{X} - X_k)}{\sqrt{2 \sqrt{L'[X_k]^{T} \Sigma X L'[X_k]}} \right)
\]

Eq 49

It can be expressed as

\[
\varepsilon = \frac{1}{2} \text{Erf} \left( \frac{M[K]}{\sqrt{2} S[K]} \right)
\]

with

\[
M[K] = L'[X_k](\bar{X} - X_k)
\]

\[
S[K] = \sqrt{L'[X_k]^{T} \Sigma X L'[X_k]}
\]

\[
L[X] = \sum_{k=1}^{np} EAD_i L G D_i P D_i [X]
\]

\[
L'[X] = \sum_{k=1}^{np} EAD_i L G D_i \frac{\partial L_i [X]}{\partial X}
\]

being \( \{X_k, \lambda_k\} \) Such a point that

\[
\begin{cases}
- (X_k - \bar{X})^{T} \Sigma^{-1} - \lambda_k L'[X_k] = 0 \\
L[X_k] - K = 0
\end{cases}
\]

Eq 50
This graph shows losses against probability, overlapping Cul d’olla a Montecarlo, with similar response obtained using more time and with a non-smoothed PDF and, therefore, not useful for optimization and fast response.

3. **Data requirements**

**Table 1: Inputs – General Parameters (Example)**

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Class</th>
<th>Origin</th>
</tr>
</thead>
<tbody>
<tr>
<td>Portfolio’s Parameters</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Spread</td>
<td>Parameter</td>
<td>Exogenous</td>
</tr>
<tr>
<td>Interest Collected (annual)</td>
<td>Parameter</td>
<td>Exogenous</td>
</tr>
<tr>
<td>Operational Cost (annual)</td>
<td>Parameter</td>
<td>Exogenous</td>
</tr>
<tr>
<td>Term (years)</td>
<td>Parameter</td>
<td>Exogenous</td>
</tr>
<tr>
<td>Early Prepayment (annual percentage)</td>
<td>Parameter</td>
<td>Exogenous</td>
</tr>
<tr>
<td>Percentage of specific coverage</td>
<td>Parameter</td>
<td>Exogenous</td>
</tr>
<tr>
<td>Generic Provision – α</td>
<td>Parameter</td>
<td>Exogenous</td>
</tr>
<tr>
<td>Generic Provision - β</td>
<td>Parameter</td>
<td>Exogenous</td>
</tr>
<tr>
<td>Recovered Past due loans (annual percentage)</td>
<td>Parameter</td>
<td>Exogenous</td>
</tr>
<tr>
<td>---------------------------------------------</td>
<td>-----------</td>
<td>-----------</td>
</tr>
<tr>
<td>Value of portfolio sold (Percentage)</td>
<td>Parameter</td>
<td>Exogenous</td>
</tr>
<tr>
<td>New X life insurance (Percentage)</td>
<td>Parameter</td>
<td>Exogenous</td>
</tr>
<tr>
<td>Provisions for Recovered loans (Percentage)</td>
<td>Parameter</td>
<td>Exogenous</td>
</tr>
<tr>
<td>Value of Recovered loans sold (Percentage)</td>
<td>Parameter</td>
<td>Exogenous</td>
</tr>
<tr>
<td>Percentage of recovered</td>
<td>Parameter</td>
<td>Exogenous</td>
</tr>
<tr>
<td><strong>Other Parameters</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Discount Rate</td>
<td>Parameter</td>
<td>Exogenous</td>
</tr>
<tr>
<td>Risk Appetite</td>
<td>Parameter</td>
<td>Exogenous</td>
</tr>
<tr>
<td>Income Tax</td>
<td>Parameter</td>
<td>Exogenous</td>
</tr>
<tr>
<td><strong>Costs</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fixed Cost (annual)</td>
<td>Parameter</td>
<td>Optional</td>
</tr>
<tr>
<td>Percentage of fixed cost (annual) variable with CPI</td>
<td>Parameter</td>
<td>Optional</td>
</tr>
<tr>
<td>Credit Investments Cost (annual)</td>
<td>Parameter</td>
<td>Optional</td>
</tr>
<tr>
<td>Financial liabilities at amortized cost (annual)</td>
<td>Parameter</td>
<td>Optional</td>
</tr>
<tr>
<td><strong>Dividends</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Payout</td>
<td>Parameter</td>
<td>Exogenous</td>
</tr>
<tr>
<td>Hurdle Rate</td>
<td>Parameter</td>
<td>Exogenous</td>
</tr>
<tr>
<td><strong>Basel Parameters</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Market Risk (Basel II)</td>
<td>Parameter</td>
<td>Exogenous</td>
</tr>
<tr>
<td>Operational Risk (Basel II)</td>
<td>Parameter</td>
<td>Exogenous</td>
</tr>
<tr>
<td>Minimum Common Equity plus capital conservation buffer (mce)</td>
<td>Parameter</td>
<td>Exogenous</td>
</tr>
</tbody>
</table>
Minimum Tier 1 Capital + Capital Conservation Buffer (mT1)  Parameter  Exogenous
Minimum Total Capital plus conservation buffer (mTP)  Parameter  Exogenous
Leverage (ap)  Parameter  Exogenous
Liquidity Coverage ratio (lcr)  Parameter  Exogenous
Net Stable Funding Ratio (nscr)  Parameter  Exogenous

Table 2: Flows entered into the simulator (Example)

<table>
<thead>
<tr>
<th>Flows</th>
<th>Class</th>
<th>Origin</th>
<th>External control</th>
</tr>
</thead>
<tbody>
<tr>
<td>Portfolio I</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>New Loans</td>
<td>Flow</td>
<td>Optimization</td>
<td>Restrictions</td>
</tr>
<tr>
<td>Depreciation</td>
<td>Flow</td>
<td>Formula</td>
<td></td>
</tr>
<tr>
<td>Prepayment</td>
<td>Flow</td>
<td>Formula</td>
<td></td>
</tr>
<tr>
<td>Delinquent credits</td>
<td>Flow</td>
<td>Formula</td>
<td></td>
</tr>
<tr>
<td>Recovered</td>
<td>Flow</td>
<td>Formula</td>
<td></td>
</tr>
<tr>
<td>Past Due Loans to Write Off</td>
<td>Flow</td>
<td>Formula</td>
<td></td>
</tr>
<tr>
<td>Defaulted Recovered</td>
<td>Flow</td>
<td>Formula</td>
<td></td>
</tr>
<tr>
<td>Portfolio for sale</td>
<td>Flow</td>
<td>Optimization</td>
<td>Restrictions</td>
</tr>
<tr>
<td>Sales of the impaired portfolio</td>
<td>Flow</td>
<td>Optimization</td>
<td>Restrictions</td>
</tr>
<tr>
<td>Deteriorated portfolio</td>
<td>Flow</td>
<td>Formula</td>
<td></td>
</tr>
<tr>
<td>Provisions</td>
<td>Flow</td>
<td>Formula</td>
<td></td>
</tr>
<tr>
<td>Failed Portfolio</td>
<td>Flow</td>
<td>Formula</td>
<td></td>
</tr>
</tbody>
</table>
Figure 5.